Dense classes of multivariate extreme value distributions

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Abstract

In this paper, we explore tail dependence modelling in multivariate extreme value distributions. The measure of dependence chosen is the scale function, which allows combinations of distributions in a very flexible way. The correspondences between the scale function and the spectral measure or the stable tail dependence function are given. Combining scale functions by simple operations, three parametric classes of laws are (re)constructed and analyzed, and resulting nested and structured models are discussed. Finally, the denseness of each of these classes is shown.

Keywords: multivariate extreme value distribution; extremal dependence; max-stable; dependence function; logistic distributions; models for multivariate extremes.

1 Introduction

Modelling the tail dependence is a main challenge in multivariate extreme value distributions. The studies in this area started with the bivariate characterizations of Tiago de Oliveira [1958], Geffroy [1958], Sibuya [1960], while multivariate representations were established by de Haan and Resnick [1977] and Pickands [1981]. In this article, the focus is on parametric and semiparametric models of extreme value distributions. This topic has been initiated by Gumbel [1960], Tiago de Oliveira [1980], Galambos [1987] and Tawn [1988]. Different reviews of parametric multivariate extreme value models are given by Coles and Tawn [1991], Joe [1997], Kotz and Nadarajah [2000] and Beirlant et al. [2004, Section 9.2.2], among others.

Our presentation will be done in terms of Fréchet margins, but other choices are possible and would lead to equivalent expressions. To illustrate these choices through the literature, one can refer e.g. to Tiago de Oliveira [1980] or Fougères et al. [2009], who worked with Gumbel marginal distributions, whereas de Haan and Resnick [1977] or Klüppelberg and May [2006] chose Fréchet margins, and Pickands [1981] or Tawn [1988] considered exponential margins.

The representation of multivariate extreme value distributions given by Pickands [1981] involves a spectral measure which underlines the main directions of dependence with a natural interpretation. Later, Huang [1992] introduced the so-called *stable tail dependence function* to model this dependence. It is entirely determined by the spectral measure, and has several properties such as homogeneity and convexity, see e.g. Beirlant et al. [2004, Section 8.2.2]. Other tools have been defined in the literature, as the Pickands [1981] dependence function A in the bivariate setting.

In the present paper, a generalization of the stable tail dependence function is introduced, that will be called the *scale function*. Both notions are close, and roughly speaking defined through the logarithm of the cumulative distribution function. The main difference between these two measures of dependence is that some information from the margins is contained in the scale function. For instance, the stable tail dependence function evaluated at each unit basis vector is equal to one, whereas the scale function at the unit basis vectors equals the corresponding margin scale. As a consequence, one may see the scale function as an unnormalized version of the stable tail dependence function. At first glance, the notion of scale function would seem perhaps unnatural. However, we have three reasons to work with this tool. The first one is that it makes it easier to construct classes of multivariate extreme value distributions by combining other ones. Renormalizing complicates these constructions and masks the essential structure. The second reason comes from the estimation point of view. When one constrains the search to guarantee a (normalized) stable tail dependence function, we force exact agreement at each unit axis vector. This is somewhat artificial, since in any practical problem, the marginals are normalized based on a sample, and thus the scaling of components is inexact. This enforced match at the margins might cause a poorer fit globally. The third reason is based on the result of de Haan [1978], who proved that for any vector following an extreme value distribution, its max-projection along any direction is univariate Fréchet, and conversely. More precisely, one can check that the scale of the univariate max-projection is given by the scale function evaluated at this direction. As a consequence, the estimation of the dependence is reduced to a sequence of univariate estimations through the estimation of max-projection scales. This topic will be addressed in a forthcoming companion paper. Note that such a method was previously used (with min-projections and under exponential margins) by Pickands [1981] and several other authors.

The main goal of this paper is to revisit already known multivariate extreme value models using their scale functions, and to define new models by combining these scale functions. The focus is on parametric and semiparametric classes that may be defined in any dimension, that are proved to be dense, and that are computationally tractable. Dealing with models in high dimension induces inference difficulties, that can be helpfully reduced by considering some parametric or semiparametric classes. A denseness property of such classes is a valuable argument to counter the idea that parametric forms are too reductive. Three classes of multivariate extreme value distributions are scrutinised: the well-known model obtained from discrete spectral measures, the generalized logistic model and the piecewise polynomial spectral density model.

The rest of this paper is laid out as follows. Section 2 introduces the scale function and gives its properties and connections with classical measures of dependence. Section 3 defines the three classes mentioned above, states useful properties of these classes, and discusses resulting nested and structured models. A simple intuitive way to quantify how close two multivariate extreme value distributions are to each other is described in Section 4, where the three classes are shown to be dense. The last section contains all the proofs.

2 Joint dependence

In this section, we examine two different ways to describe the dependence in multivariate extreme value models. The first one is the well-known spectral measure. The second is the scale function. We present their definitions in the context of unnormalized Fréchet margins.

Throughout the paper, d represents the dimension and is assumed to be greater than or equal to two. Let $\mathbf{X} = (X_1, X_2, \ldots, X_d)^T$ be a d-dimensional random vector, with multivariate extreme value distribution function denoted by G. We will assume $\mathbf{X}$ has Fréchet margins with a common shape parameter, so that

$$G_i(x_i) = \mathbb{P}(X_i \le x_i) = \exp\left(-\left(\frac{\sigma_i}{x_i - \mu_i}\right)^{\xi}\right)$$

for any $i = 1, \ldots, d$ and any $x_i > \mu_i$, where the shape parameter ξ and the scales σ_i are some positive real numbers and where the locations μ_i are real numbers.

2.1 Spectral measure

Let $\|\cdot\|$ denote any norm on $\mathbb{R}^d$ and $\mathbb{W}_+^d$ be the positive simplex in $\mathbb{R}^d$, that is to say $\mathbb{W}_+^d = \{\mathbf{x} \in \mathbb{R}_+^d, \|\mathbf{x}\| = 1\}$. According to the Representation Theorem of Pickands [1981], there exists a unique finite positive measure H on $\mathbb{W}_+^d$ such that

$$G(\mathbf{x}) = \mathbb{P}(\mathbf{X} \le \mathbf{x}) = \exp\left(-\int_{\mathbb{W}_+^d} \left(\bigvee_{i=1}^{d} \frac{w_i}{(x_i - \mu_i)^{\xi}}\right) H(d\mathbf{w})\right),$$

for any $\mathbf{x} > \boldsymbol{\mu}$. Inequalities between bold variables stand for componentwise inequalities, so that e.g. $\mathbf{x} > \boldsymbol{\mu}$ means $x_i > \mu_i$ for any $i = 1, \ldots, d$. The previous displayed formula holds true for any choice of norm on $\mathbb{R}^d$, so that the uniqueness of the measure H is with respect to this choice. Thus for a given norm one can use the notation $\mathbf{X} \sim \mathrm{Fr}(\xi, \boldsymbol{\mu}, H(\cdot))$. Note that the measure H corresponds to an unnormalized version of the spectral measure often used in the literature. Indeed, we have

$$\int_{\mathbb{W}_+^d} w_i H(d\mathbf{w}) = \sigma_i^{\xi},$$

whereas the usual normalized spectral measure $\tilde{H}$ is defined such that

$$\int_{\mathbb{W}_+^d} w_i \tilde{H}(d\mathbf{w}) = 1.$$

See Lemma 1(d) to get the relationship between H and $\tilde{H}$. The introduction of this unnormalized spectral measure is motivated by the fact that it is more convenient to construct and combine classes of multivariate extreme value distributions as we do in this paper.

When there is no ambiguity we drop the dependence on d and write just $\mathbb{W}_+$ for $\mathbb{W}_+^d$. The change of norm formula is

$$\int_{\mathbb{W}_+^{\|\cdot\|_a}} f(\mathbf{w}) H_{\|\cdot\|_a}(d\mathbf{w}) = \int_{\mathbb{W}_+^{\|\cdot\|_b}} f\left(\frac{\mathbf{w}}{\|\mathbf{w}\|_a}\right) \frac{\|\mathbf{w}\|_a}{\|\mathbf{w}\|_b} H_{\|\cdot\|_b}(d\mathbf{w}).$$

See Beirlant et al. [2004, page 264].

2.2 Scale function

We define on $\mathbb{R}_+^d$ the *scale function* as follows

$$\sigma(\mathbf{u}) = \left(\int_{\mathbb{W}_+} \left(\bigvee_{i=1}^{d} w_i u_i^{\xi}\right) H(d\mathbf{w})\right)^{1/\xi}, \tag{1}$$

which allows to write the distribution function G as

$$G(\mathbf{x}) = P(\mathbf{X} \leq \mathbf{x}) = \exp\left(-\sigma^{\xi}((\mathbf{x} - \boldsymbol{\mu})^{-1})\right), \tag{2}$$

for any $\mathbf{x} > \boldsymbol{\mu}$. This shows that the only way that the spectral measure enters into the distribution of $\mathbf{X}$ is through the scale function. As a consequence, the notation $\mathbf{X} \sim \mathrm{Fr}(\xi, \boldsymbol{\mu}, \sigma(\cdot))$ can be used equivalently to $\mathbf{X} \sim \mathrm{Fr}(\xi, \boldsymbol{\mu}, H(\cdot))$.

As already noticed in the introduction, the use of the scale function is motivated by several arguments. Since we do not have normalization constraints, the combination or construction of classes of multivariate extreme value distributions becomes simpler. Another argument comes from what we call max-projection. Assume that the locations of the margins are all equal to zero, so that we focus on multivariate extreme value distribution $\mathrm{Fr}(\xi, \boldsymbol{\mu} = \mathbf{0}, \sigma(\cdot))$. For any $\mathbf{u} = (u_1, \ldots, u_d)^T \in \mathbb{R}_+^d \setminus \{\mathbf{0}\}$, define the univariate max projection

$$M(\mathbf{u}) = \bigvee_{i=1}^{d} u_i X_i. \tag{3}$$

Then, for all $t > 0$ equation (2) implies that

$$\mathbb{P}(M(\mathbf{u}) \leq t) = \mathbb{P}(u_1 X_1 \leq t, \ldots, u_d X_d \leq t) = \exp\left(-t^{-\xi}\sigma^{\xi}(\mathbf{u})\right),$$

which means that $M(\mathbf{u})$ has a univariate Fréchet distribution $\mathrm{Fr}(\xi, \mu = 0, \sigma(\mathbf{u}))$. This shows efficiently how the dependence measure in a d-dimensional context may be reduced to a collection of one dimensional scale values (for each $\mathbf{u}$).

Remark 1.
The argument given above was shown by de Haan [1978], who proved the following equivalence: $\mathbf{X}$ *is a random vector such that max-projections (3) are univariate Fréchet for all* $\mathbf{u} \in [0, \infty)^d \setminus \{\mathbf{0}\}$ *if and only if* $\mathbf{X}$ *is multivariate Fréchet. This will be a useful tool throughout the proofs of the paper.*

Next we express the total mass of the spectral measure in terms of the scale function. The result depends on the norm chosen for the unit simplex. For the ℓ^1-norm, the simplex is $\mathbb{W}_+ = \{\mathbf{w} \in [0,1]^d, \sum_{j=1}^d w_j = 1\}$ and the total mass of the spectral measure is

$$H(\mathbb{W}_+) = \int_{\mathbb{W}_+} 1\, H(d\mathbf{w}) = \int_{\mathbb{W}_+} \sum_{j=1}^{d} w_j H(d\mathbf{w}) = \sum_{j=1}^{d} \int_{\mathbb{W}_+} w_j H(d\mathbf{w}) = \sum_{j=1}^{d} \sigma_j^{\xi}.$$

For the ℓ^∞-norm, $\mathbb{W}_+ = \{\mathbf{w} \in [0,1]^d, \max_{j=1\dots,d} w_j = 1\}$, so $\bigvee_{j=1}^d w_j = 1$ and thus

$$H(\mathbb{W}_+) = \int_{\mathbb{W}_+} 1\, H(d\mathbf{w}) = \int_{\mathbb{W}_+} \bigvee_{j=1}^{d} w_j \, H(d\mathbf{w}) = \sigma(\mathbb{1})^\xi.$$

For a function $h : \mathbb{R}^d \to \mathbb{R}$ and a set $B = \{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, d\}$, define $\partial^{|B|}h/\partial^B\mathbf{u} = \partial^k h/\partial u_{i_1} \cdots \partial u_{i_k}$. Using this notation, the following result expresses the density of a multivariate Fréchet distribution in terms of the scale function.

Proposition 1.
Let $\mathbf{X} \sim \mathrm{Fr}(\xi, \boldsymbol{\mu} = \mathbf{0}, \sigma(\cdot))$. If $\partial^d\sigma/\partial u_1 \cdots \partial u_d$ exists, then $\mathbf{X}$ has a density $g(x)$, and that density is given by

$$g(\mathbf{x}) = G(\mathbf{x}) \left(\prod_{j=1}^d x_j\right)^{-2} \sum_{\pi \in \Pi} (-1)^{|\pi|+d} \prod_{B \in \pi} \frac{\partial^{|B|}\sigma^\xi}{\partial^B \mathbf{u}}(\mathbf{x}^{-1}), \quad \mathbf{x} > 0\,,$$

where Π is the set of all partitions of $\{1, \dots, d\}$ and the product is over all of the blocks B of a partition $\pi \in \Pi$. The number $|\pi|$ denotes the number of blocks of the partition and the cardinality of each block is denoted by $|B|$.

Remark 2.
An alternative expression of the density is

$$g(\mathbf{x}) = G(\mathbf{x}) D_{12\cdots d}(x),$$

where $D_j(\mathbf{x}) := -\partial\sigma^\xi(\mathbf{x}^{-1})/\partial u_j = \xi x_j^{-2}\sigma^{\xi-1}(\mathbf{x}^{-1})\partial\sigma/\partial u_j(\mathbf{x}^{-1})$ for $j = 1, \dots, d$ and the D terms with multiple subscripts are defined recursively by $D_{12\cdots k}(\mathbf{x}) = D_{12\cdots(k-1)}(\mathbf{x})D_k(\mathbf{x}) + \partial D_{12\cdots(k-1)}(\mathbf{x})/\partial u_k$. Indeed, proceed by recursive differentiation:

$$\partial G(\mathbf{x})/\partial x_1 = \partial \exp(-\sigma^\xi(\mathbf{x}^{-1})/\partial x_1 = G(\mathbf{x})D_1(\mathbf{x})\,.$$

If $\partial^{k-1}G(\mathbf{x})/\partial x_1 \cdot \partial x_{k-1} = G(\mathbf{x})D_{12\dots(k-1)}(\mathbf{x})$ then

$$\partial^k G(\mathbf{x})/\partial x_1 \cdot \partial x_k = G(\mathbf{x})D_k(\mathbf{x}) + G(\mathbf{x})\partial D_{12\dots(k-1)}(\mathbf{x})/\partial x_k = G(\mathbf{x})D_{12\dots k}(\mathbf{x})\,.$$

2.3 Links with classical tools and properties

In the following lines, we will focus on multivariate extreme value distributions with $\boldsymbol{\mu} = \mathbf{0}$. A dependence measure often used in the literature is the so-called *stable tail dependence function* $\ell(\cdot)$ introduced by Huang [1992]. As already mentioned in the introduction, we will prefer to make use of the scale function $\sigma(\cdot)$. Indeed, in addition to the statistical arguments, the construction of new models is simplified by using $\sigma(\cdot)$ instead of $\ell(\cdot)$. The link between the two functions is given by the following relation

$$\sigma(\mathbf{u}) = \sigma(u_1, \cdots, u_d) = \ell^{1/\xi}\left((\sigma_1 u_1)^\xi, \cdots, (\sigma_d u_d)^\xi\right),$$

for each $\mathbf{u} \in \mathbb{R}_+^d$. Note in particular that if $\mathbf{X}$ has standard Fréchet margins with shape parameter $\xi = 1$, then $\sigma(\cdot)$ and $\ell(\cdot)$ are the same. Also, $V(\mathbf{u}) = \sigma^\xi(\mathbf{u})$ is the well known exponent function of de Haan and Resnick [1977]. It will sometimes be simpler to visualize the ξ-normalized version of σ defined by

$$\sigma^\star(\mathbf{u}) = \sigma^\xi(\mathbf{u}^{1/\xi}) = \int_{\mathbb{W}_+} \left(\bigvee_{i=1}^d w_i u_i\right) H(d\mathbf{w}).$$

As will be seen later, this is the scale function of the Fréchet random vector $\mathbf{X}^\xi$, which has shape parameter $\xi = 1$ and the same spectral measure H as $\mathbf{X}$ (see Lemma 1(b)). The following properties of the scale function are inherited from those of $\ell(\cdot)$ (see Beirlant et al. [2004, page 257] for a review).

($\sigma 1$) $\sigma(r\,\cdot) = r\sigma(\cdot)$, so that knowing $\sigma(\cdot)$ on $\mathbb{W}_+$ determines $\sigma(\cdot)$ everywhere;

($\sigma 2$) $\sigma(\mathbf{e}_i) = \sigma_i$ is the scale of X_i when $\mathbf{e}_i$ is the i-th standard unit vector;

($\sigma 3$) $(\sigma_1^\xi u_1^\xi \vee \cdots \vee \sigma_d^\xi u_d^\xi)^{1/\xi} \le \sigma(\mathbf{u}) \le \left(\sigma_1^\xi u_1^\xi + \cdots + \sigma_d^\xi u_d^\xi\right)^{1/\xi}$;

($\sigma 4$) $\sigma^\star(\cdot)$ is convex.

Properties ($\sigma 1 - \sigma 4$) are valid for any choice of norm on $\mathbb{R}^d$. They characterize a scale function in dimension $d = 2$, but not when $d > 2$.

Several basic operations allow us to combine multivariate extreme value distributions and stay within the class of multivariate extreme value distributions. In the following results, we describe how scale functions and spectral measures combine when this is done. While several of these facts are known, it seems useful to collect them in one place, expand the list, and see how the scale function is a useful way to represent combinations of max-stable laws. For two random vectors $\mathbf{Y}$ and $\mathbf{Z}$ the notation $\mathbf{Y} \vee \mathbf{Z}$ is used for the componentwise maximum.

Lemma 1.
Consider $\mathbf{Y}$ a d-dimensional $\mathrm{Fr}(\xi_\mathbf{Y}, \mu_\mathbf{Y}, \sigma_\mathbf{Y}(\cdot))$ and $\mathbf{Z}$ a k-dimensional $\mathrm{Fr}(\xi_\mathbf{Z}, \mu_\mathbf{Z}, \sigma_\mathbf{Z}(\cdot))$ two independent Fréchet random vectors.

(a) *Assume $d = k$ and $\mathbf{X} = \mathbf{Y} \vee \mathbf{Z}$ with $\xi_\mathbf{Y} = \xi_\mathbf{Z} = \xi$ and $\mu_\mathbf{Y} = \mu_\mathbf{Z} = \mathbf{0}$. Then, $\mathbf{X} \sim \mathrm{Fr}(\xi_\mathbf{X}, \mu_\mathbf{X}, \sigma_\mathbf{X}(\cdot))$ with $\xi_\mathbf{X} = \xi$, $\mu_\mathbf{X} = \mathbf{0}$,*

$$H_\mathbf{X} = H_\mathbf{Y} + H_\mathbf{Z}$$

and for $\mathbf{u} \in \mathbb{R}^d_+$,

$$\sigma_\mathbf{X}(\mathbf{u}) = (\sigma_\mathbf{Y}^\xi(\mathbf{u}) + \sigma_\mathbf{Z}^\xi(\mathbf{u}))^{1/\xi}.$$

(b) *Assume $\mathbf{X} = \mathbf{Y}^p$ with $\mu_\mathbf{Y} = \mathbf{0}$. Then, $\mathbf{X} \sim \mathrm{Fr}(\xi_\mathbf{X}, \mu_\mathbf{X}, \sigma_\mathbf{X}(\cdot))$ with $\xi_\mathbf{X} = \xi_\mathbf{Y}/p$, $\mu_\mathbf{X} = \mathbf{0}$,*

$$H_\mathbf{X} = H_\mathbf{Y}$$

and for $\mathbf{u} \in \mathbb{R}^d_+$,

$$\sigma_\mathbf{X}(\mathbf{u}) = (\sigma_\mathbf{Y}(\mathbf{u}^{1/p}))^p.$$

(c) *Assume $\mathbf{X} = c\mathbf{Y}$ with scalar $c > 0$. Then, $\mathbf{X} \sim \mathrm{Fr}(\xi_\mathbf{X}, \mu_\mathbf{X}, \sigma_\mathbf{X}(\cdot))$ with $\xi_\mathbf{X} = \xi_\mathbf{Y}$, $\mu_\mathbf{X} = c\mu_\mathbf{Y}$,*

$$H_\mathbf{X} = c^{\xi_\mathbf{Y}} H_\mathbf{Y}$$

and for $\mathbf{u} \in \mathbb{R}^d_+$,

$$\sigma_\mathbf{X}(\mathbf{u}) = c\,\sigma_\mathbf{Y}(\mathbf{u}).$$

(d) *Assume $\mathbf{X} = \mathbf{c}\mathbf{Y} = (c_1 Y_1, \ldots, c_d Y_d)$ with vector $\mathbf{c}$ having all $c_i > 0$ and $\mu_\mathbf{Y} = \mathbf{0}$. Then one has $\mathbf{X} \sim \mathrm{Fr}(\xi_\mathbf{X}, \mu_\mathbf{X}, \sigma_\mathbf{X}(\cdot))$ with $\xi_\mathbf{X} = \xi_\mathbf{Y}$, $\mu_\mathbf{X} = \mathbf{0}$ and for $\mathbf{u} \in \mathbb{R}^d_+$,*

$$\sigma_\mathbf{X}(u_1, \ldots, u_d) = \sigma_\mathbf{Y}(c_1 u_1, \ldots, c_d u_d).$$

If $\mathbf{Y}$ has a discrete spectral measure $H_\mathbf{Y}(\cdot) = \sum_{j=1}^m h_{\mathbf{Y},j} \delta_{\mathbf{w}_{\mathbf{Y},j}}(\cdot)$, then $\mathbf{X}$ has a discrete spectral measure $H_\mathbf{X}(\cdot) = \sum_{j=1}^m h_{\mathbf{X},j} \delta_{\mathbf{w}_{\mathbf{X},j}}(\cdot)$, where $h_{\mathbf{X},j} = \|\mathbf{v}_j\| h_{\mathbf{Y},j}$, $\mathbf{w}_{\mathbf{X},j} = \mathbf{v}_j/\|\mathbf{v}_j\|$ with $\mathbf{v}_j = (c_1^\xi w_{\mathbf{Y},j,1}, \ldots, c_d^\xi w_{\mathbf{Y},j,d})$. If $\mathbf{Y}$ has spectral density $h_\mathbf{Y}(\cdot)$ on $\mathbb{W}_+$ defined here for the ℓ_1-norm, then $H_\mathbf{X}$ is also continuous with density

$$h_\mathbf{X}(\mathbf{w}) = \|\mathbf{c}^{-\xi}\mathbf{w}\|^{-(d+1)} \left(\prod_{i=1}^d c_i\right)^{-\xi} h_\mathbf{Y}\left(\frac{\mathbf{c}^{-\xi}\mathbf{w}}{\|\mathbf{c}^{-\xi}\mathbf{w}\|}\right).$$

If the spectral measure of $\mathbf{Y}$ is a sum of a continuous and a discrete part, then the linear transformation $\mathbf{c}\mathbf{Y}$ acts on each piece separately according to the rules above.

(e) *Assume $\mathbf{X} = S^{1/\xi_\mathbf{Y}}\mathbf{Y}$ where S is a positive β-stable random variable such that $\mathbb{E}[e^{-tS}] = \exp(-t^\beta)$. Assume also that S and $\mathbf{Y}$ are independent and that $\boldsymbol{\mu}_\mathbf{Y} = \mathbf{0}$. Then $\mathbf{X} \sim \mathrm{Fr}(\xi_\mathbf{X}, \mu_\mathbf{X}, \sigma_\mathbf{X}(\cdot))$ with $\xi_\mathbf{X} = \beta\xi_\mathbf{Y}$, $\boldsymbol{\mu}_\mathbf{X} = \mathbf{0}$ and $\sigma_\mathbf{X}(\cdot) = \sigma_\mathbf{Y}(\cdot)$.*

(*f*) *Assume* $\mathbf{X} = (\mathbf{Y}^T; \mathbf{Z}^T)^T$ *with* $\xi_{\mathbf{Y}} = \xi_{\mathbf{Z}} = \xi$ *and* $\mu_{\mathbf{Y}} = \mu_{\mathbf{Z}} = \mathbf{0}$. *Then* $\mathbf{X}$ *is a* $(d+k)$*-dimensional* $\mathrm{Fr}(\xi, \boldsymbol{\mu} = \mathbf{0}, \sigma_{\mathbf{X}}(\cdot))$ *with* $\sigma^{\xi}_{\mathbf{X}}(\mathbf{u}) = \sigma^{\xi}_{\mathbf{Y}}(u_1, \dots, u_d) + \sigma^{\xi}_{\mathbf{Z}}(u_{d+1}, \dots, u_{d+k})$ *for each* $\mathbf{u} \in \mathbb{R}^{d+k}_+$ *and spectral measure* $H_{\mathbf{X}} = H_{\mathbf{Y}} \times H_{\mathbf{Z}}$.

The combination of (*b*) and (*d*) can be used for standardizing: if $\mathbf{Y} \sim \mathrm{Fr}(\xi, 0, \sigma_{\mathbf{Y}}(\cdot))$ then the random vector $\mathbf{X} = \big((Y_1/\sigma_{\mathbf{Y}}(\mathbf{e}_1))^{\xi}, \dots, (Y_d/\sigma_{\mathbf{Y}}(\mathbf{e}_d))^{\xi}\big)$ has standard Fréchet margins and its scale function is a stable tail dependence function.

The next result generalizes Lemma 1(a) when $\mathbf{Y}$ and $\mathbf{Z}$ are dependent.

Lemma 2.
Assume $\mathbf{V} = (V_1, \dots, V_{2d})^T \sim \mathrm{Fr}(\xi_{\mathbf{V}}, \mathbf{0}, \sigma_{\mathbf{V}}(\cdot))$ *and* $(\mathbf{Y}^T; \mathbf{Z}^T) := (V_1, \dots, V_d, V_{d+1}, \dots, V_{2d})$. *Then* $\mathbf{X} := \mathbf{Y} \vee \mathbf{Z} \sim \mathrm{Fr}(\xi_{\mathbf{X}}, \mathbf{0}, \sigma_{\mathbf{X}}(\cdot))$, *where* $\xi_{\mathbf{X}} = \xi_{\mathbf{V}}$ *and* $\sigma_{\mathbf{X}}(\mathbf{u}) = \sigma_{\mathbf{V}}((\mathbf{u}^T; \mathbf{u}^T)^T)$ *for each* $\mathbf{u} \in \mathbb{R}^d_+$.
Moreover, if $\mathbf{V}$ *has a discrete spectral measure* $H_{\mathbf{V}}(\cdot) = \sum_{j=1}^m h_j \delta_{\mathbf{w}_j}(\cdot)$ *on* $\mathbb{W}^{2d}_+$, *then the spectral measure for* $\mathbf{X}$ *is discrete with*

$$H_{\mathbf{X}}(\cdot) = \sum_{j=1}^{m} \tilde{h}_j \delta_{\tilde{\mathbf{w}}_j}(\cdot),$$

where $\tilde{\mathbf{w}}_j = \mathbf{t}_j / \|\mathbf{t}_j\| \in \mathbb{W}^d_+$, $\tilde{h}_j = h_j \|\mathbf{t}_j\|$ *and* $\mathbf{t}_j = (w_{j,1} \bigvee w_{j,d+1}, w_{j,2} \bigvee w_{j,d+2}, \dots, w_{j,d} \bigvee w_{j,2d})^T \in [0,1]^d$.

For a d-by-m matrix A and a vector $\mathbf{v} \in \mathbb{R}^m$, define the max product of A and $\mathbf{v}$ to be the vector in $\mathbb{R}^d$ given as follows

$$A \times_{\max} \mathbf{v} := \left(\vee_{j=1}^m a_{1j} v_j, \dots, \vee_{j=1}^m a_{dj} v_j\right)^T. \tag{4}$$

Lemma 3.
Let $\mathbf{Y}$ *be an* m*-dimensional* $\mathrm{Fr}(\xi, \mathbf{0}, \sigma_{\mathbf{Y}}(\cdot))$ *random vector and let* A *be a* d*-by-*m *matrix of nonnegative real numbers. Then* $\mathbf{X} = A \times_{\max} \mathbf{Y}$ *is a* d*-dimensional* $\mathrm{Fr}(\xi, \mathbf{0}, \sigma_{\mathbf{X}}(\cdot))$ *with* $\sigma_{\mathbf{X}}(\mathbf{u}) = \sigma_{\mathbf{Y}}(A^T \times_{\max} \mathbf{u})$ *for each* $\mathbf{u} \in \mathbb{R}^d_+$. *If* $\mathbf{Y}$ *has a discrete spectral measure* $H_{\mathbf{Y}}(\cdot) = \sum_{j=1}^n h_j \delta_{\mathbf{w}_j}(\cdot)$ *on* $\mathbb{W}^m_+$, *then* $\mathbf{X}$ *has the discrete spectral measure* $H_{\mathbf{X}}(\cdot) = \sum_{j=1}^n \widetilde{h}_j \delta_{\widetilde{\mathbf{w}}_j}(\cdot)$ *on* $\mathbb{W}^d_+$, *where* $\widetilde{\mathbf{w}}_j = \mathbf{t}_j / \|\mathbf{t}_j\|$, $\widetilde{h}_j = h_j \|\mathbf{t}_j\|$, *and* $\mathbf{t}_j = (\vee_{i=1}^m w_{j,i} a^{\xi}_{1i}, \vee_{i=1}^m w_{j,i} a^{\xi}_{2i} \dots, \vee_{i=1}^m w_{j,i} a^{\xi}_{di})^T \in \mathbb{R}^d$.

Remark 3.
The operation given in Lemma 3 is another way to obtain several results from Lemma 1 and 2. More specifically,

- *Taking* $A = diag(c_1, \dots, c_d)$ *gives* $A \times_{\max} \mathbf{Y} = (c_1 Y_1, \dots, c_d Y_d)$ *and Lemma 3 implies Lemma 1(d).*
- *Letting* $\mathbf{V}$ *be* $2d$*-dimensional and* $A = (I; I)$ *be a* d*-by-*$2d$ *matrix,* $A \times_{\max} \mathbf{V}$ *gives Lemma 2.*
- *If* $\mathbf{X}^T = (\mathbf{Y}^T; \mathbf{Z}^T)$ *where* $\mathbf{Y}$ *and* $\mathbf{Z}$ *are independent, setting* $A = (I; I)$ *be a* d*-by-*$2d$ *matrix, then* $A \times_{\max} \mathbf{X} = \mathbf{Y} \vee \mathbf{Z}$, *and Lemma 3 combined with Lemma 1(f) implies Lemma 1(a).*
- *More general combinations are possible. As an illustration, consider two independent Fréchet random vectors with same shape parameter, namely* $\mathbf{Y}$ *of dimension* d *and* $\mathbf{Z}$ *of dimension* k. *Combining Lemma 1(f) and Lemma 3 shows that for positive constants* c_i, *the random vector defined by*

$$\left(\begin{array}{cccc|ccccc} c_1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 & c_3 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & c_4 \\ 0 & c_5 & \cdots & 0 & 0 & c_6 & c_7 & \cdots & 0 \end{array}\right) \times_{\max} \begin{pmatrix} \mathbf{Y} \\ \mathbf{Z} \end{pmatrix} = \begin{pmatrix} c_1 Y_1 \\ c_2 Y_2 \vee c_3 Z_1 \\ (\vee_{i=1}^d Y_i) \vee (\vee_{j=1}^k Z_j) \\ c_4 Z_k \\ c_5 Y_2 \vee c_6 Z_2 \vee c_7 Z_3 \end{pmatrix}$$

  follows a 5-dimensional Fréchet distribution.

Since both triplets (ξ, μ, H) and (ξ, μ, σ) characterize the multivariate extreme value distribution G, one can wonder about the link between these representations. If the spectral measure H is known, then the scale $\sigma(\cdot)$ is known by (1). Conversely, knowing $\sigma(\cdot)$ determines the spectral measure H, but there is no explicit formula for H in general. However, in specific cases (e.g the discrete spectral model) one can recognize the form of $\sigma(\cdot)$ and identify H.

In the bivariate case, we introduce the unnormalized Pickands' function by

$$B(t) = \sigma(1-t, t) \ , \tag{5}$$

for each $t \in [0,1]$, while the ξ-normalized definition is

$$B^\star(t) = \sigma^\star(1-t, t). \tag{6}$$

In this case, we identify the simplex $\mathbb{W}_+$ with the interval $[0,1]$. The next result states the extension of Beirlant et al. [2004, Equation (8.47)] to our unnormalized framework. Note that it uses arguments mainly contained in Pickands [1981, Theorem 3.1].

Lemma 4.
Assume that $\mathbf{X}$ *is* $\mathrm{Fr}(\xi, \boldsymbol{\mu} = \mathbf{0}, H(\cdot))$ *or* $\mathrm{Fr}(\xi, \boldsymbol{\mu} = \mathbf{0}, \sigma(\cdot))$ *for* $d = 2$*. Then for* $t \in (0,1)$

$$H([0,t]) = B^{\star\prime}(t) + \sigma_1^\xi \ ,$$

where $B^{\star\prime}$ *should be interpreted as the right derivative of the function defined by (6). The point masses are*

$$H(\{0\}) = B^{\star\prime}(0) + \sigma_1^\xi \ ,$$
$$H(\{1\}) = -B^{\star\prime}(1) + \sigma_2^\xi.$$

Before ending the section, we present a way to simplify the computation of the density of differentiable multivariate Fréchet models by reducing to the case $\xi = 1$.

Lemma 5.
Let $\mathbf{Y} \sim \mathrm{Fr}(\xi, \boldsymbol{\mu} = \mathbf{0}, \mathrm{H}_\mathbf{Y})$ *and assume the existence of a density* $g_\mathbf{Y}(\cdot)$*. Then the random vector* $\mathbf{X} = \mathbf{Y}^\xi$ *is* $\mathrm{Fr}(1, \boldsymbol{\mu} = \mathbf{0}, \mathrm{H}_\mathbf{Y})$ *and it has density for* $\mathbf{x} > \mathbf{0}$ *given by*

$$g_\mathbf{X}(\mathbf{x}) = \xi^{-d} \left(\prod_{k=1}^d x_k \right)^{(1/\xi)-1} g_\mathbf{Y}(\mathbf{x}^{1/\xi}) \ .$$

3 Classes of multivariate extreme value distributions

In this section, we describe several classes of multivariate extreme value distributions. Among these models only one is non differentiable; two can be easily simulated and all of them lead to parametric or semi-parametric forms for the scale function. We also study the closure property of these models under the operations introduced in Lemmas 1, 2 and 3. Analogous results for general multivariate extreme value distributions can be found e.g. in [Resnick, 1987, page 253] and [Beirlant et al., 2004, page 267].

3.1 Discrete spectral measures

Max-stable distributions with discrete spectral measures have been consider by multiple authors (see e.g. Deheuvels [1983] or Einmahl et al. [2011]), and this section is mostly a collection of previously known facts. At the end of this subsection, we discuss some reasons why it is worth examining this class. Let m be a positive integer, $\{h_1, \dots, h_m\}$ some non-negative real numbers and $\mathbf{w}_j = (w_{j,1}, \dots, w_{j,d})^T$ for $j = 1, \dots, m$ elements of the simplex $\mathbb{W}_+$ for a given norm $\|\cdot\|$. If the spectral measure is discrete, say $H(\cdot) = \sum_{j=1}^m h_j \delta_{\mathbf{w}_j}(\cdot)$, then the scale functions are, for $\mathbf{u} \in \mathbb{R}_+^d$,

$$\sigma(\mathbf{u}) = \left(\sum_{j=1}^m h_j \left(\vee_{i=1}^d w_{j,i} u_i^\xi \right) \right)^{1/\xi}$$

and

$$\sigma^\star(\mathbf{u}) = \sum_{j=1}^m h_j \left(\vee_{i=1}^d w_{j,i} u_i \right) .$$

The model based on discrete spectral measure is the most tractable and works in any dimension. In Proposition 4 of Section 4 we give a simple proof of the fact that this class is dense.

The vertices of the piecewise linear scale function have value

$$\sigma(\mathbf{w}_k) = \left(\sum_{j=1}^{m} h_j \left(\vee_{i=1}^{d} w_{j,i} w_{k,i}^{\xi} \right) \right)^{1/\xi},$$

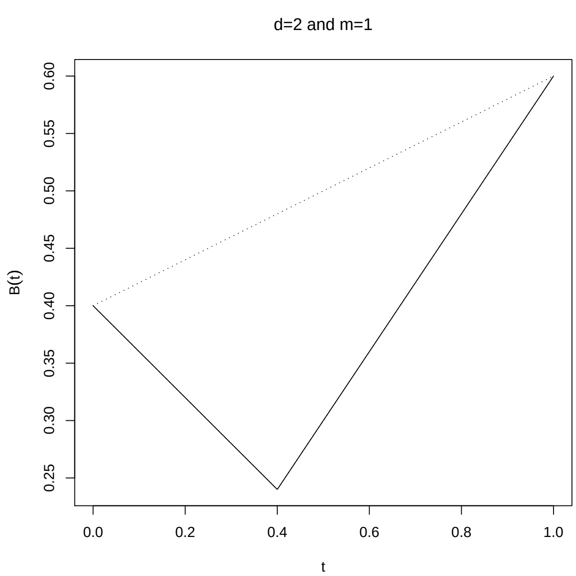
for each $k = 1, \cdots, m$, which can be written as $\sigma(\mathbf{w}_k)^{\xi} = \sum_{j=1}^{m} h_j M_{k,j}$ for $M_{k,j} := \vee_{i=1}^{d} w_{j,i} w_{k,i}^{\xi}$. This directly gives a linear system for the powered-values of the scale functions at the vertices in terms of the weights:

$$\begin{pmatrix} \sigma(\mathbf{w}_1)^{\xi} \\ \sigma(\mathbf{w}_2)^{\xi} \\ \vdots \\ \sigma(\mathbf{w}_m)^{\xi} \end{pmatrix} = M \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{pmatrix}$$

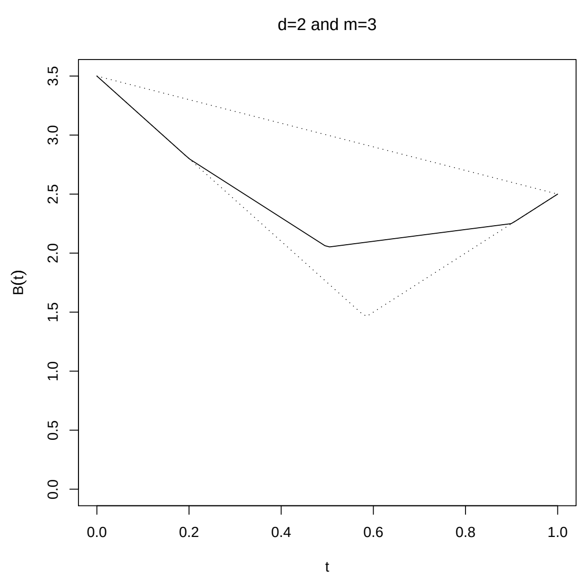
where M is the m-by-m matrix $M = [M_{k,j}]$. If the matrix M is invertible, knowing $\sigma^{\xi}(\cdot)$ at the vertices completely determines the discrete spectral measure.

Lemma 6.
The class of multivariate Fréchet distributions with discrete spectral measures is closed under the operations (a), (b), (c), (d) and (f) of Lemma 1, and under the operations of Lemmas 2 and 3.

As an example of multivariate Fréchet distribution with discrete spectral measure, consider a single point mass, thus $\sigma^{\star}(\mathbf{u}) = h_1(\vee_{i=1}^{d} w_{1,i} u_i)$. In a bivariate setting, $\sigma^{\star}(\cdot)$ restricted to the unit simplex is the V-shaped function $B^{\star}(t) = \sigma^{\star}(1-t, t)$. The function $B^{\star}$ has one vertex at the point $t = w_{1,1}$, see Figure 1 (a). When the dimension is three or more the graph of the function is still a V-shaped function, i.e. a flat-sided cone with vertex at a point. When there are m point masses, the function $\sigma^{\star}(\cdot)$ is the sum of m V-shaped functions. In particular, if the dimension is $d = 2$ it is piecewise linear with vertices at $(\mathbf{w}_j, \sigma^{\star}(\mathbf{w}_j))$ and end points $(\mathbf{e}_i, \sigma^{\star}(\mathbf{e}_i))$, $i = 1, \ldots, d$. See Figure 1 (b).



(a)



(b)

Figure 1: The ξ-normalized Pickands' function $B^{\star}$ with upper and lower bounds from condition (σ_3).
(a) $d = 2$, $m = 1$, $\mathbf{w}_1 = (0.4, 0.6)^T$ and $h_1 = 1$.
(b) $d = 2$, $m = 3$, $\mathbf{w}_1 = (0.2, 0.8)^T$, $\mathbf{w}_2 = (0.5, 0.5)^T$, $\mathbf{w}_3 = (0.9, 0.1)^T$, $h_1 = 1$, $h_2 = 3$ and $h_3 = 2$.

One nice feature of the discrete model for the multivariate extreme value distribution is that it is straightforward to simulate. Let $\mathbf{Z} = (Z_1, \ldots, Z_m)^T$ be composed by m independent and identically distributed univariate Fréchet$(\xi, \mu = 0, \sigma = 1)$ and let $A = [a_{ij}]$ be a d-by-m matrix with non-negative entries. Then it can be seen easily that

$$\mathbf{X} := A \times_{\max} \mathbf{Z} = \left(\vee_{j=1}^m a_{1j} Z_j, \vee_{j=1}^m a_{2j} Z_j, \ldots, \vee_{j=1}^m a_{dj} Z_j\right)^T \tag{7}$$

is Fr$(\xi, \boldsymbol{\mu} = \mathbf{0}, H(\cdot))$ with discrete spectral measure H having mass $h_j = \|a_{\cdot j}^{\xi}\|$ at point $\mathbf{w}^j = a_{\cdot j}^{\xi}/h_j$, where $a_{\cdot j}$ denotes the j's column of A. See also Stephenson [2003, Theorem 1] for similar considerations. The discrete spectral measure is thus a model that can be constructed from a sequence of m independent factors $Z_1, \ldots, Z_m$. In applications, this model arises as the attractor of the so-called "factor model", often used in practice (see for example Einmahl et al. [2008, 2011] for more details). One can e.g. think of d lines of insurance, say $X_1, \ldots, X_d$, where each line may be affected in different ways by storms, the severity of which is modeled by Z_i, for $i = 1, \ldots, m$. It may also be possible in certain situations to find a small number of factors $Z_1, \ldots, Z_m$ that give a good description of a high dimensional problem, i.e. $m << d$.

The discrete spectral measure model allows a nice understanding of the main directions of dependence. As shown in the previous section, one can frequently get closed form expressions for combinations of max-stable vectors with discrete spectral measures. This allows one to explicitly see how some operation affects the spectral measure, which may not be clear for a general spectral measure.

The scale function $\sigma(\cdot)$ will not be differentiable if H is a discrete spectral measure with mass on the interior of $\mathbb{W}_+$, so the density $g(\cdot)$, obtained by differentiating (2), will not exist. The non-differentiability of the scale function causes the cumulative distribution function G to have "creases" along the rays where there are point masses, which is why the density does not exist. Hence discrete spectral measures correspond to non-smooth distributions, which may not be appropriate for some problems. The next two classes of models lead to smooth scale functions, and hence they will have a density $g(\cdot)$.

3.2 Generalized logistic mixtures

Several models have been defined combining positive stable distributions and extreme value distributions. For earlier results, see Coles and Tawn [1991, §4.2], as well as Hougaard [1986], Crowder [1989], Tawn [1990]. More recently, Fougères et al. [2009] unified the results in the previous papers and used them to construct structured models, e.g. max-stable time series. The key point is to produce dependent Fréchet distributions by mixing independent Fréchet components with independent sum-stable scales. In the latter paper, the focus is on the fact that in these models, both conditional and unconditional distributions are extreme value distributions. The following result allows more general dependence in the terms of the mixture distribution. Note that the results Fougères et al. [2009] were mainly stated in terms of Gumbel margins and it assumed a restricted form for the sum-stable vector; here we use Fréchet margins and an arbitrary positive sum-stable vector.

In the following, a positive multivariate stable distribution with index α is the law of a positive random vector $\mathbf{S} = (S_1, \ldots, S_d)^T$ with Laplace transform

$$\mathbb{E}[e^{-<\mathbf{u},\mathbf{S}>}] = \exp(-c_\alpha \gamma^\alpha(\mathbf{u})) \ , \mathbf{u} \in \mathbb{R}_+^d$$

where $c_\alpha = \sec(\pi\alpha/2)$ and

$$\gamma^\alpha(\mathbf{u}) = \int_{\mathbb{S}_+} < \mathbf{u}, \mathbf{s} >^\alpha \Lambda(d\mathbf{s}) \ . \tag{8}$$

In the previous display, $\alpha \in (0,1)$ and $\mathbb{S}_+$ is the first orthant of the unit sphere in the Euclidean norm, and Λ denotes a positive and finite measure on $\mathbb{S}_+$. We will say that $\mathbf{S} = (S_1, \ldots, S_d)^T$ is a positive α-stable random vector with sum-stable spectral measure Λ. See Samorodnitsky and Taqqu [1994, Proposition 1.2.12]. Note that each margin of $\mathbf{S}$ is a positive α-stable random variable with $\mathbb{E}[e^{-tS_j}] = \exp(-c_\alpha t^\alpha \gamma^\alpha(\mathbf{e}_j))$. Moreover, note that any positive linear combination of components of $\mathbf{S}$ is a univariate positive α-stable random variable.

Theorem 1.
Let $Z_1, \ldots, Z_d$ be independent and identically distributed univariate Fréchet $(\xi, \mu = 0, \sigma = 1)$, and $\mathbf{Z} = (Z_1, \ldots, Z_d)^T$. Let $\alpha \in (0,1)$ and $\mathbf{S} = (S_1, \ldots, S_d)^T$ be a positive α-stable random vector with sum-stable spectral measure Λ that is independent of $Z_1, \ldots, Z_d$. Then the random vector

$$\mathbf{X} := \mathbf{S}^{1/\xi} \cdot \mathbf{Z} = (S_1^{1/\xi} Z_1, \ldots, S_d^{1/\xi} Z_d)^T \tag{9}$$

is $\mathrm{Fr}(\alpha\xi, \boldsymbol{\mu} = \mathbf{0}, \sigma(\cdot))$ *with scale function for* $\mathbf{u} \in \mathbb{R}_+^d$

$$\sigma^{\alpha\xi}(\mathbf{u}) = c_\alpha \gamma^\alpha(\mathbf{u}^\xi), \tag{10}$$

where the right hand side is given by (8).

The class of multivariate extreme value distributions with scale function defined by (10) will be called a *generalized logistic mixture* or *generalized logistic model.* Several properties may be pointed out. It is clear that it is available for any dimension. This model is differentiable and we state the general expression of its density in Proposition 2. Moreover we prove that it is a dense subset (possibly with only a few terms) in Theorem 5. The Gumbel case of Fougères et al. [2009] is obtained if we take logarithms of each component. In the Fréchet setting, the stable terms S_i are random scales of the original Z_i. In the Gumbel case we have

$$V_i := \log X_i = W_i + \frac{1}{\xi} \log S_i,$$

for $i = 1, \ldots, d$, where $W_i = \log Z_i$ are independent and Gumbel distributed. Several interpretations of this Gumbel shifted model are given in Fougères et al. [2009].

Remark 4.
It is interesting to examine the result stated in Theorem 1 in terms of the margins. Consider one of the components, say $Z_1 S_1^{1/\xi}$. *The tail behaviors of the two terms are* $\mathbb{P}(Z_1 > x) \sim x^{-\xi}$ *and* $\mathbb{P}(S_1^{1/\xi} > x) \sim x^{-\alpha\xi}$. *By Breiman's lemma (see e.g. Resnick [2007, section 7.3.2]), the heavier tail dominates in the product, so we get* X_1 *in the domain of attraction of a Fréchet distribution with index* $\alpha\xi$, *since* $\alpha < 1$. *The fact that we get exactly a Fréchet law is a pleasing algebraic fact. If we started with terms with the same tails, but not exactly of the same type, the product would be in the domain of attraction of a Fréchet law with index* $\alpha\xi$.

By construction, one can simulate generalized logistic vectors $\mathbf{X}$ as soon as one can simulate positive α-stable random vectors $\mathbf{S}$. We know how to simulate positive α-stable random vector $\mathbf{S}$ with discrete spectral measure Λ (a dense subset); see Modarres and Nolan [1994]. If the stable random vector $\mathbf{S}$ has a discrete sum-stable spectral measure $\Lambda(\cdot) = \sum_{j=1}^m \lambda_j \delta_{\mathbf{s}_j}(\cdot)$, then the scale function of Fréchet $\mathbf{X}$ is, for $\mathbf{u} \in \mathbb{R}_+^d$,

$$\sigma(\mathbf{u}) = \left(c_\alpha \sum_{j=1}^m \lambda_j \langle \mathbf{u}^\xi, \mathbf{s}_j \rangle^\alpha \right)^{1/(\alpha\xi)}. \tag{11}$$

The generalized logistic model with discrete sum-stable spectral measure Λ recovers several models intensively studied in the literature. The best known classes are the logistic model and the mixed model, respectively introduced by Gumbel [1960] and Tawn [1988]. Note that our definitions may slightly differ from those of the literature. This comes from the normalization. When the margins are normalized, mass points are added on the axes (corresponding to independent components) so that the scale of each margin is one.

- In the case of one mass only, the above reduces to $\sigma(\mathbf{u}) = (\sum_{i=1}^d (u_i v_i)^\xi)^{1/\xi}$ where $v_i = c_\alpha \lambda_1 s_{1,i}^{1/\xi}$. For $\xi = 1/\alpha$, we identify the scale function of an (unnormalized) simple asymmetric logistic distribution. The particular mass $\lambda_1 = d^\alpha / c_\alpha$ and location $\mathbf{s}_1 = 1/d(1, \ldots, 1)^T$ give the symmetric logistic case: $\sigma(\mathbf{u}) = (\sum_i u_i^{1/\alpha})^\alpha$.
- When several point masses are present in the measure Λ ($m > 1$), the scale function (11) is a mixture of asymmetric logistic terms.
- When we take a more general sum-stable spectral measure Λ, e.g. with a continuous density, we obtain a larger class of asymmetric logistic mixtures.

This subclass of multivariate Fréchet distributions is stable under some transformations. More precisely, one gets the following result.

Lemma 7.
The class of generalized logistic distributions is closed under the operations (a), (b), (c), (d) and (e) of Lemma 1.

Remark 5.
In all generality the generalized logistic mixtures is not closed under the operation (f) of Lemma 1. However, consider $\mathbf{X} = (\mathbf{Y}^T; \mathbf{Z}^T)^T$ *with* $\xi_{\mathbf{Y}} = \xi_{\mathbf{Z}} = \xi$ *and two generalized logistic mixtures given by* $\mathbf{Y} = \mathbf{S}^{1/\xi_U} \cdot \mathbf{U}$ *and* $\mathbf{Z} = \mathbf{T}^{1/\xi_V} \cdot \mathbf{V}$*. In the special case where* $\xi_{\mathbf{U}} = \xi_{\mathbf{V}}$ *and* $\alpha_{\mathbf{S}} = \alpha_{\mathbf{T}}$*,* $(\mathbf{S}^T; \mathbf{T}^T)^T$ *is a* $(d+k)$*-dimensional positive* $\alpha_{\mathbf{S}}$*-stable random vector with stable spectral measure* $\Lambda_{\mathbf{S}} \times \Lambda_{\mathbf{T}}$*, so that* $\mathbf{X}$ *is a generalized logistic mixture.*

The generalized logistic model is differentiable, as already pointed out in Coles and Tawn [1991, §4.1] for the asymmetric logistic mixture model. The form of the density function is specified in the next result.

Proposition 2.
Under the assumption of Theorem 1, $\mathbf{X}$ *is a continuous random vector with cumulative distribution function* $G(\mathbf{x}) = \exp\left(-c_\alpha \gamma^\alpha(\mathbf{x}^{-\xi})\right)$ *for* $\mathbf{x} > \mathbf{0}$*. Its density is*

$$g(\mathbf{x}) = \left\{ \sum_{\pi \in \Pi} (-1)^{|\pi|+d} \prod_{B \in \pi} \frac{\partial^{|B|} I(\mathbf{x})}{\partial^B \mathbf{x}} \right\} \times G(\mathbf{x}) \,, \tag{12}$$

with

$$\frac{\partial^{|B|} I(\mathbf{x})}{\partial^B \mathbf{x}} := c_\alpha \frac{\alpha!}{(\alpha - |B|)!} \xi^{|B|} \int_{\mathbb{S}_+} \left(\langle \mathbf{x}^{-\xi}, \mathbf{s} \rangle^{\alpha - |B|} \prod_{i \in B} s_i x_i^{-\xi - 1} \right) \Lambda(ds) \,,$$

where in the expression (12) the sum is over Π *the set of all partitions of* $\{1, \ldots, d\}$ *and the product is over all of the blocks* B *of a partition* $\pi \in \Pi$*. The number* $|\pi|$ *denotes the number of blocks of the partition and the cardinality of each block is denoted by* $|B|$*.*

The previous result is now illustrated for the discrete sum-stable spectral measure: $\Lambda(\cdot) = \sum_{j=1}^m \lambda_j \delta_{\mathbf{s}_j}(\cdot)$, where $\mathbf{s}_j = (s_{j1}, \ldots, s_{jd})^T \in \mathbb{S}_+$. In this case, for each $\mathbf{x} > \mathbf{0}$, one has

$$\frac{\partial^{|B|} I(\mathbf{x})}{\partial^B \mathbf{x}} = c_\alpha \frac{\alpha!}{(\alpha - |B|)!} \xi^{|B|} \sum_{j=1}^m \lambda_j \langle \mathbf{x}^{-\xi}, \mathbf{s}_j \rangle^{\alpha - |B|} \prod_{i \in B} s_{ji}\, x_i^{-\xi - 1}.$$

We restrict in the following to the 2-dimensional model of generalized logistic mixtures in order to establish the relation between the stable spectral measure Λ and the max-stable spectral measure H, or its density denoted h.

Proposition 3.
Let $\mathbf{X}$ *be a bivariate generalized logistic random vector defined by (9). The density of the associated spectral measure* H *is given, for any* $t \in [0,1]$*, by*

$$h(t) = c_\alpha (1/\alpha - 1)(t(1-t))^{1/\alpha - 2} \int_{\mathbb{S}_+} ((1-t)^{1/\alpha} s_1 + t^{1/\alpha} s_2)^{\alpha - 2} s_1 s_2 \Lambda(d\mathbf{s}).$$

When Λ is discrete with a single point mass at $\mathbf{s} = (s_1, s_2)$ with mass λ we obtain

$$h(t) = c_\alpha \lambda (1/\alpha - 1)(t(1-t))^{1/\alpha - 2} ((1-t)^{1/\alpha} s_1 + t^{1/\alpha} s_2)^{\alpha - 2} s_1 s_2.$$

This corresponds to the bivariate asymmetric logistic model of Tawn [1988] with $\psi_i^{1/\alpha} = c_\alpha s_i \lambda$ for $i = 1, 2$.

It does not appear to be possible to allow dependence of the Fréchet terms $Z_1, \ldots, Z_d$ in the Theorem 1 in general. However, we present now a particular case, when $\mathbf{S}$ is totally dependent, where dependence in the Z_i's is allowed. Let $\mathbf{Z} = (Z_1, \ldots, Z_d)^T$ be a multivariate Fréchet $\mathrm{Fr}(\xi, \boldsymbol{\mu} = \mathbf{0}, \sigma_{\mathbf{Z}}(\cdot))$. Let $\mathbf{S} = S\mathbf{v}$ where S is a positive α-stable random variable such that $\mathbb{E}[e^{-tS}] = \exp(-t^\alpha)$ and $\mathbf{v} = (v_1, \ldots, v_d)^T$ is a deterministic vector of $\mathbb{S}_+$. Then $\mathbf{X}$ defined by (9) is $\mathrm{Fr}(\alpha\xi, \boldsymbol{\mu} = \mathbf{0}, \sigma_{\mathbf{X}}(\cdot))$ with, for each $\mathbf{u} \in \mathbb{R}_+^d$,

$$\sigma_{\mathbf{X}}^{\alpha\xi}(\mathbf{u}) = \sigma_{\mathbf{Z}}^{\alpha\xi}(v_1^{1/\xi} u_1, \ldots, v_d^{1/\xi} u_d).$$

This results from the combination of the operations (d) and (e) of Lemma 1.

3.3 Piecewise polynomial spectral densities

Klüppelberg and May [2006] consider polynomials defined on the whole simplex as models for the bivariate Pickands' function. In contrast, here we consider spectral measures $H(\cdot)$ in arbitrary dimensions that are absolutely continuous with densities $h(\cdot)$ that are piecewise polynomial. This model has five attractive properties. First, using a piecewise definition allows one to spread mass locally, not forcing a global description. Second, it has tractable computational properties. Third, we can estimate it even in high dimension (if the number of pieces is not too large). Fourth, it gives a dense family in arbitrary dimensions. And finally, this model is smooth in the sense that its distributions have densities. An open question is to find an interpretation of this class, and to find a way to simulate from it.

In two dimensions, one can explicitly compute the scale function for a piecewise polynomial spectral density. We identify again the simplex with the interval $[0,1]$. We start with a monomial w^k on an interval. Let $k \in (-1,\infty)$ (not necessarily an integer), $0 \le a < b \le 1$ and define for $u \in [0,1]$

$$\alpha_k(u;a,b) = \int_a^b [(1-u)w \vee u(1-w)]w^k dw.$$

Lemma 8.
The function α_k has the following expression

$$\alpha_k(u;a,b) \quad = \quad \begin{cases} c_1(1-u) & u \le a \\ c_2 u^{k+2} - c_3 u + c_4 & a < u < b \\ c_5 u & u > b, \end{cases}$$

where

$$\begin{aligned}
c_1 &= c_1(k,a,b) = \frac{b^{k+2} - a^{k+2}}{k+2}, \\
c_2 &= c_2(k) = \frac{1}{k+1} - \frac{1}{k+2}, \quad c_3 = c_3(k,a,b) = c_1(k,a,b) + \frac{a^{k+1}}{k+1}, \quad c_4 = c_4(k,b) = \frac{b^{k+2}}{k+2}, \\
c_5 &= c_5(k,a,b) = \frac{b^{k+1} - a^{k+1}}{k+1} - c_1.
\end{aligned}$$

Let us make some comments on the previous coefficients. The first remark is that all the c_i's are positive. Also, there are relationships among these parameters so that the function $\alpha_k(\cdot;a,b)$ is continuous:

$$\begin{cases} \alpha_k(a^-;a,b) = \alpha_k(a^+;a,b) = c_1(1-a) \\ \alpha_k(b^-;a,b) = \alpha_k(b^+;a,b) = c_5 b \end{cases}$$

and differentiable:

$$\begin{cases} \alpha_k'(a^-;a,b) = \alpha_k'(a^+;a,b) = -c_1 \\ \alpha_k'(b^-;a,b) = \alpha_k'(b^+;a,b) = c_5. \end{cases}$$

The second derivative of $\alpha_k(\cdot;a,b)$ does not exist at the join points a and b, whatever the value of k. Visually, $\alpha_k(\cdot;a,b)$ is a cone, with straight line segments to the left of a and to the right of b, and a rounded vertex given by a power function of degree $(k+2)$ in the interval $[a,b]$. Note that if $k \ge 0$, the scale function is smooth and hence the corresponding bivariate extreme value distribution has a density.
Using these terms as building blocks, we can explicitly evaluate the scale function for a piecewise polynomial spectral density. Let $0 \le a_1 < a_2 < \cdots < a_{m+1} \le 1$ and $w \in [0,1]$. If a piecewise polynomial spectral density is given by

$$h(w) = \sum_{j=1}^m p_j(w) 1_{(a_j,a_{j+1}]}(w) = \sum_{j=1}^m \left(\sum_{k=0}^N b_{k,j} w^k \right) 1_{(a_j,a_{j+1}]}(w) \ ,$$

then the scale function is, for each $(u_1,u_2) \in \mathbb{R}_+^2$,

$$\sigma(u_1,u_2) = \left(\sum_{j=1}^m \left\{ \sum_{k=0}^N b_{k,j} \left(u_1^\xi + u_2^\xi \right) \alpha_k \left(\frac{u_2^\xi}{u_1^\xi + u_2^\xi}; a_j, a_{j+1} \right) \right\} \right)^{1/\xi} .$$

Figure 2 illustrate piecewise linear spectral densities h and the corresponding ξ-normalized Pickands' functions $B^\star(t)=\sum_{j=1}^m \left\{\sum_{k=0}^N b_{k,j}\alpha_k\left(t;a_j,a_{j+1}\right)\right\}$.

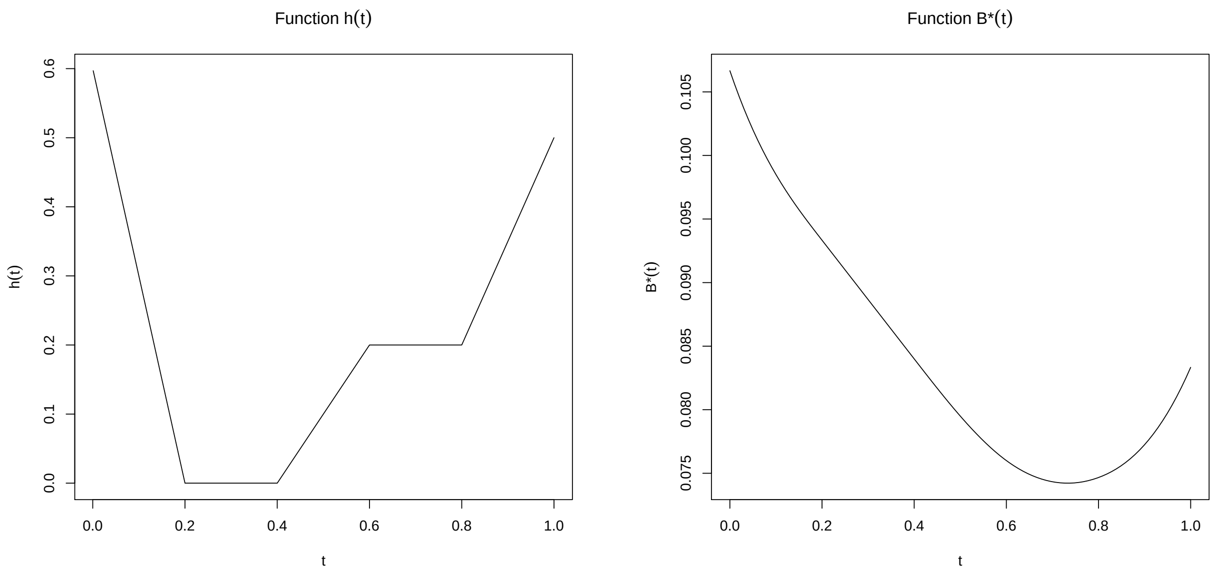


Figure 2: A piecewise linear spectral density (left) and the corresponding Pickands' function $B^\star$ (right) for $m=5$, $N=1$, $a=(0,0.2,0.4,0.6,0.8,1)^T$, $b_{0,\cdot}=(0.6,0,-0.4,0.2,-1)^T$ and $b_{1,\cdot}=(-3,0,1,0,1.5)^T$. Note that $B^*(t)$ is piecewise cubic on (0,0.2), (0.4,0.6) and (0.8,1); quadratic on (0.6,0.8); and linear on (0.2,0.4).

The explicit formula for the scale function gives an explicit formula for the distribution function. This in turn gives an explicit formula for the density $g(\mathbf{x})$. This can be done directly, or using Proposition 1. In the case $\xi=1$ and $\boldsymbol{\mu}=\mathbf{0}$, the expressions are:

$$\begin{aligned}
G(x_1,x_2) &= \exp(-\sigma(x_1^{-1},x_2^{-1}))\\
g(x_1,x_2) &= \frac{\partial^2 G}{\partial x_1\partial x_2}(\mathbf{x}) = \frac{G(\mathbf{x})}{x_1^2x_2^2}\left[\frac{\partial\sigma}{\partial u_1}(x_1^{-1},x_2^{-1})\frac{\partial\sigma}{\partial u_2}(x_1^{-1},x_2^{-1}) - \frac{\partial^2\sigma}{\partial u_1\partial u_2}(x_1^{-1},x_2^{-1})\right]
\end{aligned}$$

$$\begin{aligned}
\text{where}\qquad \sigma(u_1,u_2) &= \sum_{j=1}^m\sum_{k=0}^N b_{k,j}\,(u_1+u_2)\,\alpha_k\left(\frac{u_2}{u_1+u_2};a_j,a_{j+1}\right)\;,\\
\frac{\partial\sigma}{\partial u_1}(u_1,u_2) &= \sum_{j=1}^m\sum_{k=0}^N b_{k,j}\left[\alpha_k\left(\frac{u_2}{u_1+u_2};a_j,a_{j+1}\right) - \frac{u_2}{u_1+u_2}\alpha_k'\left(\frac{u_2}{u_1+u_2};a_j,a_{j+1}\right)\right]\;,\\
\frac{\partial\sigma}{\partial u_2}(u_1,u_2) &= \sum_{j=1}^m\sum_{k=0}^N b_{k,j}\left[\alpha_k\left(\frac{u_2}{u_1+u_2};a_j,a_{j+1}\right) + \frac{u_1}{u_1+u_2}\alpha_k'\left(\frac{u_2}{u_1+u_2};a_j,a_{j+1}\right)\right]\;,\\
\frac{\partial^2\sigma}{\partial u_1\partial u_2}(u_1,u_2) &= -\frac{u_1u_2}{(u_1+u_2)^3}\sum_{j=1}^m\sum_{k=0}^N b_{k,j}\alpha_k''\left(\frac{u_2}{u_1+u_2};a_j,a_{j+1}\right)\;.
\end{aligned}$$

Figure 3 shows a simple example of these formulas.

The previous definition may be extended to higher dimensions. Any polynomial p of degree less or equal to N can be written as

$$p(\mathbf{w}) = \sum_{k_1+\ldots+k_d\le N} b_{\mathbf{k}}\mathbf{w}^{\mathbf{k}}$$

where $\mathbf{k}=(k_1,\ldots,k_d)$ is a multi-index of non-negative integers and $\mathbf{w}^{\mathbf{k}}=w_1^{k_1}w_2^{k_2}\ldots w_d^{k_d}$. Let $\Delta_1,\ldots,\Delta_m$ be a partition of $\mathbb{W}_+\subset\mathbb{R}^d$ by convenient sets, say $(d-1)$-simplices. We define the piecewise polynomial

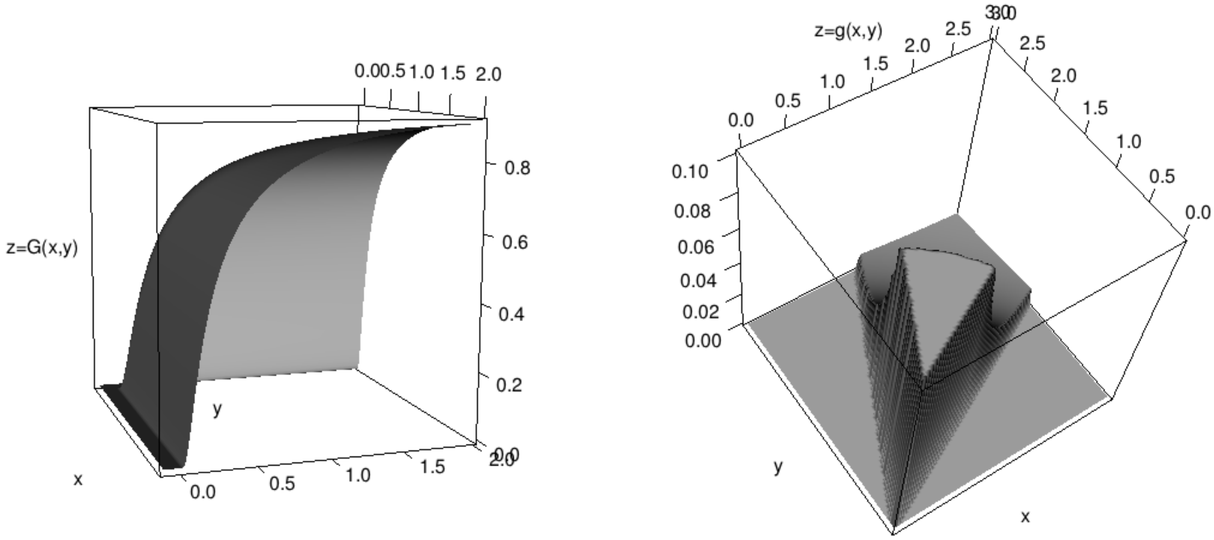


Figure 3: The cumulative distribution function G for a piecewise constant spectral density $h(\mathbf{w}) = \mathbf{1}_{[1/3,2/3]}(\mathbf{w})$ (left) and its corresponding density g (right). Note that the vertical scale is cut at height 0.1 for display purposes.

spectral density by

$$h(\mathbf{w}) = \sum_{j=1}^{m} p_j(\mathbf{w}) 1_{\Delta_j}(\mathbf{w}).$$

We require that the polynomial p_j is non-negative on Δ_j. The multivariate Fréchet distribution with piecewise polynomial spectral density h corresponds to the scale function

$$\sigma(\mathbf{u}) = \left(\sum_{j=1}^{m} \int_{\Delta_j} \left(\bigvee_{i=1}^{d} u_i^{\xi} w_i \right) p_j(\mathbf{w}) d\mathbf{w} \right)^{1/\xi}, \quad \mathbf{u} \in \mathbb{R}_+^d ,$$

with the ξ-normalized version given by

$$\sigma^\star(\mathbf{u}) = \sum_{j=1}^{m} \int_{\Delta_j} \left(\bigvee_{i=1}^{d} u_i w_i \right) p_j(\mathbf{w}) d\mathbf{w}.$$

Lemma 9.
The class of multivariate Fréchet distributions with piecewise polynomial spectral density is closed under the operations (a), (b), (c), (d) and (f) of Lemma 1.

For a vector $\mathbf{u}$ of $\mathbb{R}_+^d$, let us introduce the set $T_\ell(\mathbf{u}) = \left\{ \mathbf{w} \in \mathbb{W}_+, \vee_{i=1}^{d} u_i w_i = u_\ell w_\ell \right\}$ for $\ell = 1, \ldots, d$. The T_ℓ's are closed $(d-1)$-dimensional polytopes, overlapping only along edges that form $(d-2)$ dimensional sets. These sets cover $\mathbb{W}_+$, and since we are only considering continuous spectral measures, these intersections have no mass, and we can regard $\{T_\ell, \ell = 1, \ldots, d\}$ as a partition of $\mathbb{W}_+$. (If an exact partition is required when dealing with a non-continuous H, one can eliminate the overlap: for $\ell > 1$ replace T_ℓ with $T_\ell - \cup_{j<\ell} T_j$.) We have

$$\sigma^\star(\mathbf{u}) = \sum_{j=1}^{m} \sum_{\ell=1}^{d} u_\ell \left(\sum_{k_1+\ldots+k_d \leq N} b_{j,\mathbf{k}} \int_{\Delta_j \cap T_\ell(\mathbf{u})} w_\ell \mathbf{w}^{\mathbf{k}} d\mathbf{w} \right).$$

We remark that $\Delta_j \cap T_\ell(\mathbf{u})$ is a $(d-1)$-dimensional polytope in $\mathbb{W}_+$. By triangulation techniques one may obtain a partition of $\Delta_j \cap T_\ell(\mathbf{u})$ into $(d-1)$-dimensional simplices. With some further simplifications, numerical computation is then possible since exact formulas exist for integrating a polynomial on a simplex. See for instance Baldoni et al. [2011, Corollary 20]. The details of the computational method will be presented in a related paper dealing with the estimation procedure for these models.

3.4 Structured and nested models

In this section we discuss classes of multivariate extreme value distributions that are structured in some way. Rather than considering $\mathbf{X}$ as a general d-dimensional random vector, we assume that there is some specified way the joint distribution is defined, generally through some fixed structure among a subset of components. In the model building process, this fixed structure will be naturally imposed by specific types of dependence, like a temporal or a spatial dependence, as we will consider in the following paragraphs. Throughout this section, the Z_i's are independent and identically distributed univariate Fréchet variables.

Structured models

First, we revisit Fougères et al. [2009], where classes of stable mixtures were considered, and extend some ideas. As mentioned above, the framework of *temporal dependence* is common in practice, and one is often interested in modelling a dependence on the "past" using a time series. In the linear setting, there is a well developed theory of ARMA and more general models; here we describe extreme value time series models based on discrete spectral measures and generalized logistic models.

For a univariate time series model with a discrete spectral measure, define $X_t = \vee_{k=0}^m a_k Z_{t-k}$, for $t \in \{1, \ldots, d\}$. This can be equivalently written as in (7) in terms of a $(m+d)$-vector of Z_i's, and a d-by-$(m+d)$ matrix of coefficients A that has a band structure. Note that in this case, the discrete spectral measure H of the series $(X_1, \ldots, X_d)$ is supported on $(m+1)$ dimensional faces of the d-dimensional unit simplex, with a fixed structure. One can extend this concept for a multivariate time series with a discrete spectral measure: this is essentially the M4 process as introduced by Smith and Weissman [1996], see for example Zhang and Smith [2004].

Introducing time series models via generalized logistic distributions was done in an univariate framework in Fougères et al. [2009], with the stable terms were the sum of a finite number of "past" terms: $X_t = S_t Z_t$, where $S_t = \sum_{k=0}^m a_k T_{t-k}$ and T_i are independent and identically distributed univariate positive stable. Note that in this case the stable spectral measure is discrete and is supported on $(m+1)$ dimensional faces of the unit sphere, again with a fixed structure. In matrix form, this model can be presented as follows. Consider the stable discrete spectral measure $\Lambda(\cdot) = \sum_{j=0}^m \lambda_j \delta_{\mathbf{s}_j}(\cdot)$. It corresponds to the spectral measure of the stable vector $\mathbf{S} = P\mathbf{T}$ for $\mathbf{T} = (T_0, \ldots, T_m)^T$ and $P = [P_{ij}]$ a matrix of size $d \times (m+1)$, where $P_{\cdot j}$ its jth-column satisfies $P_{\cdot j} = \lambda_j^{1/\alpha} \mathbf{s}_j$. Note that one can also extend this construction to multivariate time series.

Let us now consider the framework of *spatial dependence*: A similar idea can be used in spatial models, where only the nearby components are dependent. Let $\mathbf{t} = (t_1, \ldots, t_k)^T$ be an index of locations on a lattice T^k. Let N_0 be a fixed neighborhood of 0. For the discrete spectral measure case, define $X_{\mathbf{t}} = \vee_{\mathbf{j} \in N_0} a_{\mathbf{j}} Z_{\mathbf{t}+\mathbf{j}}$ for some positive constants $a_{\mathbf{j}}$. This can again be rewritten as in (7). Note that similarly, the generalized logistic case was defined in Fougères et al. [2009]: $X_{\mathbf{t}} = S_{\mathbf{t}} Z_{\mathbf{t}}$, where $S_{\mathbf{t}} = \sum_{\mathbf{j} \in N_0} a_{\mathbf{j}} Z_{\mathbf{t}+\mathbf{j}}$.

Another construction of spatial models is to build *distributions on a graph*: Suppose G is a graph with nodes $\{v_i : i = 1, \ldots, m\}$ and adjacency matrix A: $a_{i,j} = 1$ if node i and j are connected, otherwise $a_{i,j} = 0$; we do not require A to be symmetric. We will write $i \sim j$ if i is connected to j. We can define a multivariate Fréchet distribution on G in several ways. For a discrete spectral model, define $X_i = \vee_{j \sim i} b_{i,j} Z_j$. A similar definition can be used for the generalized logistic distributions, considering $X_i = S_i Z_i$, where $S_i = \sum_{j \sim i} b_{i,j} Z_j$. An application of this type of models in an environmental framework is to look at a river system, where one models water flow at multiple locations. In this situation, the nodes can be the measuring sites, and node i is connected to node j if i is immediately downstream from j. Here the generalized logistic model may be appropriate, as the height at one point is likely connected to the sum of factors from the upstream sites.

Nested models

Nested models are ones where there is a chain of classes of models $\{A_n, n = 1, \ldots, N\}$ (for $N \leq \infty$) with $A_n \subset A_{n+1}$. This means that if a spectral measure $H \in A_n$, then $H \in A_{n+1}$. Equivalently, this can be stated in terms of scale functions: if $\sigma(\cdot) \in A_n$ then $\sigma(\cdot) \in A_{n+1}$. This may be useful for model selection, where some criteria is used to decide whether to use a more complex model. For each of the three classes discussed in the paper, there are two natural ways to do this. The first way is

- For discrete spectral measures, let A_n be the set of finite discrete spectral measures with n-point masses.

- For generalized logistic models, let A_n be the set of generalized logistic models with stable random vector $\mathbf{S}$ having support with n-point masses.

- For piecewise polynomial models, let A_n be the set of max-stable distributions arising from piecewise polynomial spectral densities of degree n.

The second way is more specific:

- For discrete spectral measures, let $\{B_n\}$ be sets of points in $\mathbb{W}_+$ with $B_n \subset B_{n+1}$ and A_n the set of finite discrete spectral measures with support B_n. For example the B_n's could be successive refinements of a grid.

- For generalized logistic models, let $\{B_n\}$ be a nested set of points in $\mathbb{S}_+$. Then define A_n as the generalized logistic models with stable random vector $\mathbf{S}$ having support B_n.

- For piecewise polynomial models, let $\{B_n\}$ be a nested collection of partitions of $\mathbb{W}_+$ and define A_n as the set of max-stable distributions arising from piecewise polynomial spectral densities on B_n.

4 Metrics for multivariate extreme value distributions

Throughout this section we consider multivariate Fréchet distributions $\mathrm{Fr}(\xi, \boldsymbol{\mu} = \mathbf{0}, \sigma(\cdot))$ satisfying the assumption

$$\sigma_0 := \inf_{\mathbf{u} \in \mathbb{W}_+} \sigma(\mathbf{u}) > 0. \tag{13}$$

This infimum measures how close the distribution is to singular: σ_0 is zero if and only if one or more of the margin scales are zero, which is equivalent to have a distribution concentrated on a lower dimensional subspace of $\mathbb{R}^d$. If $\xi = 1$ and all $\sigma_i > 0$, then two lower bounds are

$$\sigma(\mathbf{u}) \geq \left\| \left(\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_d} \right) \right\|^{-1} \geq \frac{\min_{i=1,\ldots,d} \sigma_i}{\|\mathbf{1}\|}.$$

The first inequality comes from the left side of ($\boldsymbol{\sigma}\mathbf{3}$) – see beginning of Section 2.3 – which is minimized at $\mathbf{u}^\star = \left(\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_d}\right) / \left\|\left(\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_d}\right)\right\|$. The second inequality follows from the first and $\sigma_j \geq \min_{i=1,\ldots,d} \sigma_i$. As a consequence, a sufficient condition to get (13) is to assume that all the margins are non trivial or equivalently that the distribution has full dimension. In terms of the spectral measure H, $\sigma_i > 0$ if and only if $H(\{\mathbf{w} : w_i > 0\}) > 0$. Thus $\sigma_0 > 0$ if and only if $H(\{\mathbf{w} : w_i > 0\}) > 0$ for all $i = 1, \ldots, d$.

The ideas in this section are adapted from the sum-stable case in Nolan [2010]. Our main result is the following. It says that if two multivariate Fréchet distributions have similar scale functions, then their cumulative distribution functions are uniformly close.

Theorem 2.
Let $\mathbf{X} \sim \mathrm{Fr}(\xi, \boldsymbol{\mu}_{\mathbf{X}} = \mathbf{0}, \sigma_{\mathbf{X}}(\cdot))$ and $\mathbf{Y} \sim \mathrm{Fr}(\xi, \boldsymbol{\mu}_{\mathbf{Y}} = \mathbf{0}, \sigma_{\mathbf{Y}}(\cdot))$ both satisfying (13) and with respective cumulative distribution functions $G_{\mathbf{X}}$ and $G_{\mathbf{Y}}$. If

$$\sup_{\mathbf{u} \in \mathbb{W}_+} |\sigma_{\mathbf{X}}^\xi(\mathbf{u}) - \sigma_{\mathbf{Y}}^\xi(\mathbf{u})| \leq \delta$$

for some $0 < \delta < \sigma_0^\xi$, then

$$\sup_{\mathbf{x} \in \mathbb{R}^d} |G_{\mathbf{X}}(\mathbf{x}) - G_{\mathbf{Y}}(\mathbf{x})| \leq \frac{2\delta}{\sigma_0^\xi} \,,$$

where $\sigma_0 = \min\{\inf_{\mathbf{u} \in \mathbb{W}_+} \sigma_{\mathbf{X}}(\mathbf{u}), \inf_{\mathbf{u} \in \mathbb{W}_+} \sigma_{\mathbf{Y}}(\mathbf{u})\}$.

The next result rephrases the preceding result in terms of spectral measures: if two Fréchet distributions have similar spectral measures, then their distributions are close. More precisely, consider a norm $\|\cdot\|$ on $\mathbb{R}^d$. Let $H_{\mathbf{X}}$ and $H_{\mathbf{Y}}$ be two spectral measures on $\mathbb{W}_+$ associated to this norm. Define the extended Prokorov metric π^* by

$$\pi^*(H_{\mathbf{X}}, H_{\mathbf{Y}}) = \pi\left(\frac{H_{\mathbf{X}}}{H_{\mathbf{X}}(\mathbb{W}_+)}, \frac{H_{\mathbf{Y}}}{H_{\mathbf{Y}}(\mathbb{W}_+)}\right) + |H_{\mathbf{X}}(\mathbb{W}_+) - H_{\mathbf{Y}}(\mathbb{W}_+)|\,,$$

where π is the Prokorov metric on the space of probability measures on $\mathbb{W}_+$. In particular, $\pi^*(H_{\mathbf{X}}, H_{\mathbf{Y}})$ will be small when $H_{\mathbf{X}}$ and $H_{\mathbf{Y}}$ have total mass about the same and their normalizations to probability measures are close in the Prokorov metric.

Theorem 3.
Let $\mathbf{X} \sim \mathrm{Fr}(\xi, \boldsymbol{\mu}_{\mathbf{X}} = \mathbf{0}, H_{\mathbf{X}}(\cdot))$ and $\mathbf{Y} \sim \mathrm{Fr}(\xi, \boldsymbol{\mu}_{\mathbf{Y}} = \mathbf{0}, H_{\mathbf{Y}}(\cdot))$ both satisfying (13) and with respective cumulative distribution functions $G_{\mathbf{X}}$ and $G_{\mathbf{Y}}$. If

$$\pi^*(H_{\mathbf{X}}, H_{\mathbf{Y}}) \leq \delta$$

for some $0 < \delta < \sigma_0^{\xi}/(K^2(1+K^2))$, then

$$\sup_{\mathbf{x} \in \mathbb{R}^d} |G_{\mathbf{X}}(\mathbf{x}) - G_{\mathbf{Y}}(\mathbf{x})| \leq 2K^2(1+K^2)\delta/\sigma_0^{\xi} \,,$$

where $\sigma_0 = \min\{\inf_{\mathbf{u} \in \mathbb{W}_+} \sigma_{\mathbf{X}}(\mathbf{u}), \inf_{\mathbf{u} \in \mathbb{W}_+} \sigma_{\mathbf{Y}}(\mathbf{u})\}$ and $K = K_{\infty, \|\cdot\|}$ satisfies $\|\mathbf{x}\|_\infty \leq K\|\mathbf{x}\|$ for any $\mathbf{x} \in \mathbb{R}^d$.

We now show that the models based on discrete spectral measures, defined in Section 3.1, are dense.

Proposition 4.
Let $\mathbf{X} \sim \mathrm{Fr}(1, \boldsymbol{\mu} = \mathbf{0}, H(\cdot))$ satisfying (13) and with cumulative distribution function G. For any $\epsilon > 0$, there exists a cumulative distribution function G_{disc}, associated to a multivariate Fréchet with discrete spectral measure with a finite number of point masses, uniformly close to G:

$$|G(\mathbf{x}) - G_{\mathrm{disc}}(\mathbf{x})| \leq \epsilon \qquad \textit{for all } \mathbf{x} \in \mathbb{R}^d.$$

Note that this result has been proved in the minima setting by Deheuvels [1983]. The next proposition presents the denseness property for the generalized logistic mixtures, studied in Section 3.2. A similar result when $d = 2$ has been independently obtained by H. Rootzén, A. Rudvik, and C. Borrell (private communication).

Proposition 5.
Let $\mathbf{X} \sim \mathrm{Fr}(1, \boldsymbol{\mu} = \mathbf{0}, H(\cdot))$ satisfying (13) and with cumulative distribution function G. For any $\epsilon > 0$, there exists a cumulative distribution function $G_{\log}$, associated to a generalized logistic mixture, uniformly close to G:

$$|G(\mathbf{x}) - G_{\log}(\mathbf{x})| \leq \epsilon \qquad \textit{for all } \mathbf{x} \in \mathbb{R}^d.$$

Remark 6.
Let emphasize that the generalized logistic distribution $G_{\log}$ defined in Proposition 5 depends on ϵ. Indeed, the proof shows that for every ϵ, $G_{\log}$ is constructed using (9) in terms of a positive multivariate stable distribution with index $\alpha = \alpha(\epsilon)$ and sum-stable discrete spectral measure $\Lambda = \Lambda(\epsilon)$ with a finite number of point masses.

Finally, the equivalent formulation for the piecewise polynomial spectral densities, introduced in Section 3.3, is given.

Proposition 6.
Let $\mathbf{X} \sim \mathrm{Fr}(1, \boldsymbol{\mu} = \mathbf{0}, H(\cdot))$ satisfying (13) and with cumulative distribution function G. For any $\epsilon > 0$, there exists a cumulative distribution function G_{pp}, associated to a piecewise polynomial spectral measure, uniformly close to G:

$$|G(\mathbf{x}) - G_{\mathrm{pp}}(\mathbf{x})| \leq \epsilon \qquad \textit{for all } \mathbf{x} \in \mathbb{R}^d.$$

The preceding propositions offer three different approximations of any multivariate Fréchet distribution, and therefore of any multivariate extreme value distribution after well chosen marginal transformations. In practice, there is no abstract reason to choose one of these models over another. It is unlikely that one will be able to distinguish between these classes with real data, unless there is a massive data set. However, the choice of a model can be based on some physical understanding of the situation where the data is obtained, or on arguments such as parsimony, existence of a density, etc. For example, in higher dimensions, it may be preferable to use a generalized logistic model with a few terms that gives a smooth model, than a discrete spectral measure with many terms.

5 Proofs

Proof of Proposition 1: We use the differentiation of a function of the form $\exp(\phi(\mathbf{x}))$. More precisely, we apply the fact that

$$\frac{\partial^d}{\partial x_1 \dots \partial x_d} \exp(\phi(\mathbf{x})) = \exp(\phi(\mathbf{x})) \sum_{\pi \in \Pi} \prod_{B \in \pi} \frac{\partial^{|B|} \phi(\mathbf{x})}{\partial^B \mathbf{x}} .$$

In our case $\phi(\mathbf{x}) = -\sigma^\xi(\mathbf{x}^{-1})$ so that

$$\frac{\partial^{|B|} \phi(\mathbf{x})}{\partial^B \mathbf{x}} = \left(\prod_{i \in B} \frac{1}{x_i^2} \right) (-1)^{1+|B|} \frac{\partial^{|B|} \sigma^\xi}{\partial^B \mathbf{x}} (\mathbf{x}^{-1})$$

which allows to conclude since $\prod_{B \in \pi} (-1)^{1+|B|} = (-1)^{|\pi|+d}$.

Proof of Lemma 1: Throughout this proof, $\mathbf{u} = (u_1, \dots, u_d)^T \in \mathbb{R}_+^d \setminus \{\mathbf{0}\}$ is arbitrary and $t > 0$.
(a) Following Remark 1, it suffices to prove that any max-projection as defined by (3) is univariate Fréchet. Consider the random variable $\vee_{i=1}^d u_i X_i$. By independence of $\mathbf{Y}$ and $\mathbf{Z}$, we have

$$\begin{aligned}
\mathbb{P}(\vee_{i=1}^d u_i X_i \leq t) &= \mathbb{P}\left(\vee_{i=1}^d u_i \max(Y_i, Z_i) \leq t\right) \\
&= \mathbb{P}\left(Y_1 \leq t/u_1, \dots, Y_d \leq t/u_d, Z_1 \leq t/u_1, \dots, Z_d \leq t/u_d\right) \\
&= \mathbb{P}\left(Y_1 \leq t/u_1, \dots, Y_d \leq t/u_d\right) \mathbb{P}\left(Z_1 \leq t/u_1, \dots, Z_d \leq t/u_d\right) \\
&= \exp\left(-\sigma_{\mathbf{Y}}^\xi(u_1 t^{-1}, \dots, u_d t^{-1})\right) \exp\left(-\sigma_{\mathbf{Z}}^\xi(u_1 t^{-1}, \dots, u_d t^{-1})\right) \\
&= \exp\left(-t^{-\xi} \left\{\sigma_{\mathbf{Y}}^\xi(\mathbf{u}) + \sigma_{\mathbf{Z}}^\xi(\mathbf{u})\right\}\right) .
\end{aligned}$$

The previous equality also implies that $\xi_{\mathbf{X}} = \xi$ and $\sigma_{\mathbf{X}}^\xi(\cdot) = \sigma_{\mathbf{Y}}^\xi(\cdot) + \sigma_{\mathbf{Z}}^\xi(\cdot)$. The equality $H_{\mathbf{X}} = H_{\mathbf{Y}} + H_{\mathbf{Z}}$ follows easily from the integral representation of the scale function given by (1).
(b) Again consider a max-projection $\vee_{i=1}^d u_i X_i$. We have

$$\begin{aligned}
\mathbb{P}(\vee_{i=1}^d u_i X_i \leq t) &= \mathbb{P}\left(\vee_{i=1}^d u_i Y_i^p \leq t\right) \\
&= \mathbb{P}\left(Y_1 \leq (t/u_1)^{1/p}, \dots, Y_d \leq (t/u_d)^{1/p}\right) \\
&= \exp\left(-\sigma_{\mathbf{Y}}^\xi(u_1^{1/p} t^{-1/p}, \dots, u_d^{1/p} t^{-1/p})\right) \\
&= \exp\left(-t^{-\xi/p} \sigma_{\mathbf{Y}}^\xi(\mathbf{u}^{1/p})\right) .
\end{aligned}$$

We deduce that $\mathbf{X}$ is a multivariate Fréchet random vector with $\xi_{\mathbf{X}} = \xi/p$ and $\sigma_{\mathbf{X}}^{\xi/p}(\mathbf{u}) = \sigma_{\mathbf{Y}}^\xi(\mathbf{u}^{1/p})$. Again the equality $H_{\mathbf{X}} = H_{\mathbf{Y}}$ is a direct consequence of the combination of (1) with the relation of the scale functions.
(c) Let $\vee_{i=1}^d u_i X_i$ be a max-projection. For simplicity, we write ξ for $\xi_{\mathbf{Y}}$. We have

$$\begin{aligned}
\mathbb{P}(\vee_{i=1}^d u_i X_i \leq t) = \mathbb{P}\left(\vee_{i=1}^d u_i c Y_i \leq t\right) = \mathbb{P}\left(Y_1 \leq (t/\{cu_1\}), \dots, Y_d \leq (t/\{cu_d\})\right) = \exp\left(-\sigma_{\mathbf{Y}}^\xi(cu_1 t^{-1}, \dots, cu_d t^{-1})\right) \\
= \exp\left(-t^{-\xi} c^\xi \sigma_{\mathbf{Y}}^\xi(\mathbf{u})\right) .
\end{aligned}$$

It yields $\mathbf{X}$ is a multivariate Fréchet random vector with $\xi_{\mathbf{X}} = \xi$ and $\sigma_{\mathbf{X}}^\xi(\cdot) = c^\xi \sigma_{\mathbf{Y}}^\xi(\cdot)$. We obtain $H_{\mathbf{X}} = c^\xi H_{\mathbf{Y}}$ by combining (1) with the relation between the scale functions.
(d) Let $\vee_{i=1}^d u_i X_i$ be a max-projection. For simplicity, we write ξ for $\xi_{\mathbf{Y}}$. Then

$$\begin{aligned}
\mathbb{P}(\vee_{i=1}^d u_i X_i \leq t) &= \mathbb{P}\left(\vee_{i=1}^d u_i c_i Y_i \leq t\right) = \mathbb{P}\left(Y_1 \leq t/\{u_1 c_1\}, \dots, Y_d \leq t/\{u_d c_d\}\right) \\
&= \exp\left(-\sigma_{\mathbf{Y}}^\xi(u_1 c_1 t^{-1}, \dots, u_d c_d t^{-1})\right) = \exp\left(-t^{-\xi} \sigma_{\mathbf{Y}}^\xi(\mathbf{uc})\right) .
\end{aligned}$$

This implies that $\mathbf{X}$ is a multivariate Fréchet random vector with $\xi_{\mathbf{X}} = \xi$ and $\sigma_{\mathbf{X}}(\mathbf{u}) = \sigma_{\mathbf{Y}}(\mathbf{uc})$.
We first focus on the discrete case. From the equalities given above,

$$\sigma_{\mathbf{X}}^\xi(\mathbf{u}) = \sigma_{\mathbf{Y}}^\xi(\mathbf{uc}) = \sum_{j=1}^m h_{\mathbf{Y},j} \bigvee_{i=1}^d w_{\mathbf{Y},j,i} (c_i u_i)^\xi = \sum_{j=1}^m h_{\mathbf{Y},j} \bigvee_{i=1}^d (w_{\mathbf{Y},j,i} c_i^\xi) u_i^\xi,$$

which allows us to conclude by identifying the sum in the exponent with $\sigma_{\mathbf{X}}^{\xi}(\mathbf{u}) = \sum_{j=1}^{m} h_{\mathbf{X},j} \vee_{i=1}^{d} w_{\mathbf{X},j,i} u_i^{\xi}$. For the continuous case, substituting $\mathbf{w} = \mathbf{v}\mathbf{c}^{-\xi}/\|\mathbf{v}\mathbf{c}^{-\xi}\|$ shows

$$\begin{aligned}
\sigma_{\mathbf{X}}^{\xi}(\mathbf{u}) &= \sigma_{\mathbf{Y}}^{\xi}(\mathbf{c}\mathbf{u}) = \int_{\mathbb{W}_+} \bigvee_{i=1}^{d} w_i (c_i u_i)^{\xi} h_{\mathbf{Y}}(\mathbf{w}) d\mathbf{w} = \int_{\mathbb{W}_+} \bigvee_{i=1}^{d} (w_i c_i^{\xi}) u_i^{\xi} h_{\mathbf{Y}}(\mathbf{w}) d\mathbf{w} \\
&= \int_{\mathbb{W}_+} \bigvee_{i=1}^{d} (v_i/\|\mathbf{v}\mathbf{c}^{-\xi}\|) u_i^{\xi} h_{\mathbf{Y}}(\mathbf{v}\mathbf{c}^{-\xi}/\|\mathbf{v}\mathbf{c}^{-\xi}\|) J(\mathbf{v}) d\mathbf{v} \\
&= \int_{\mathbb{W}_+} \bigvee_{i=1}^{d} v_i u_i^{\xi} h_{\mathbf{Y}}(\mathbf{v}\mathbf{c}^{-\xi}/\|\mathbf{v}\mathbf{c}^{-\xi}\|) J(\mathbf{v})/\|\mathbf{v}\mathbf{c}^{-\xi}\| d\mathbf{v}
\end{aligned}$$

where $J(\mathbf{v})$ is the Jacobean of the transformation. Thus $\mathbf{X}$ has a spectral density given by

$$h_{\mathbf{X}}(\mathbf{v}) = h_{\mathbf{Y}}(\mathbf{v}\mathbf{c}^{-\xi}/\|\mathbf{v}\mathbf{c}^{-\xi}\|) J(\mathbf{v})/\|\mathbf{v}\mathbf{c}^{-\xi}\|.$$

Calculations as in Coles and Tawn [1991, Theorem 2] show $J(\mathbf{v}) = \|\mathbf{v}\mathbf{c}^{-\xi}\|^{-d} \prod_{i=1}^{d} c_i^{-\xi}$, so

$$h_{\mathbf{X}}(\mathbf{v}) = h_{\mathbf{Y}}(\mathbf{v}\mathbf{c}^{-\xi}/\|\mathbf{v}\mathbf{c}^{-\xi}\|) \, \|\mathbf{v}\mathbf{c}^{-\xi}\|^{-(d+1)} \prod_{i=1}^{d} c_i^{-\xi}.$$

(e) For simplicity, we write ξ instead of $\xi_{\mathbf{Y}}$. The distribution of $\vee_{i=1}^{d} u_i X_i$ is

$$\begin{aligned}
\mathbb{P}(\vee_{i=1}^{d} u_i X_i \le t) &= \mathbb{P}(u_1 Y_1 S^{1/\xi} \le t, \dots, u_d Y_d S^{1/\xi} \le t) \\
&= \mathbb{P}(Y_1 \le t S^{-1/\xi}/u_1, \dots, Y_d \le t S^{-1/\xi}/u_d) \\
&= \mathbb{E}_S \left[\mathbb{P}(Y_1 \le t s^{-1/\xi}/u_1, \dots, Y_d \le t s^{-1/\xi}/u_d) | S = s \right] \\
&= \mathbb{E}_S \left[\exp(-\sigma_{\mathbf{Y}}^{\xi}(u_1 s^{1/\xi}/t, \dots, u_d s^{1/\xi}/t)) | S = s \right] \\
&= \mathbb{E}_S \left[\exp(-t^{-\xi} s \sigma_{\mathbf{Y}}^{\xi}(\mathbf{u})) | S = s \right] \\
&= \mathbb{E} \left[\exp(-t^{-\xi} S \sigma_{\mathbf{Y}}^{\xi}(\mathbf{u})) \right] = \exp(-t^{-\beta\xi} \sigma_{\mathbf{Y}}^{\beta\xi}(\mathbf{u})) \; .
\end{aligned}$$

This allows us to conclude that $\xi_{\mathbf{X}} = \beta\xi$ and $\sigma_{\mathbf{X}}(\cdot) = \sigma_{\mathbf{Y}}(\cdot)$.
(f) By independence of $\mathbf{Y}$ and $\mathbf{Z}$, write

$$\begin{aligned}
P(\mathbf{X} \le \mathbf{x}) &= P(Y_1 \le x_1, \dots, Y_d \le x_d, Z_1 \le x_{d+1}, \dots, Z_k \le x_{d+k}) \\
&= P(Y_1 \le x_1, \dots, Y_d \le x_d) P(Z_1 \le x_{d+1}, \dots, Z_k \le x_{d+k}) \\
&= \exp(-\sigma_{\mathbf{Y}}^{\xi}(1/x_1, \dots, 1/x_d)) \exp(-\sigma_{\mathbf{Z}}^{\xi}(1/x_{d+1}, \dots, 1/x_{d+k})) \\
&= \exp(-\left[\sigma_{\mathbf{Y}}^{\xi}(1/x_1, \dots, 1/x_d) + \sigma_{\mathbf{Z}}^{\xi}(1/x_{d+1}, \dots, 1/x_{d+k}) \right]).
\end{aligned}$$

Hence $\sigma_{\mathbf{X}}^{\xi}(\mathbf{u}) = \sigma_{\mathbf{Y}}^{\xi}(u_1, \dots, u_d) + \sigma_{\mathbf{Z}}^{\xi}(u_{d+1}, \dots, u_{d+k})$. To obtain the announced form of H_X, just proceed by identification through the following set of equalities:

$$\begin{aligned}
\int_{\mathbb{W}_+^{d+k}} \left(\bigvee_{i=1}^{d+k} u_i^{\xi} w_i \right) H_{\mathbf{X}}(d\mathbf{w}) = \sigma_{\mathbf{X}}^{\xi}(\mathbf{u}) &= \sigma_{\mathbf{Y}}^{\xi}(u_1, \dots, u_d) + \sigma_{\mathbf{Z}}^{\xi}(u_{d+1}, \dots, u_{d+k}) \\
&= \int_{\mathbb{W}_+^{d}} \left(\bigvee_{i=1}^{d} u_i^{\xi} s_i \right) H_{\mathbf{Y}}(d\mathbf{s}) + \int_{\mathbb{W}_+^{k}} \left(\bigvee_{i=1}^{k} u_{d+i}^{\xi} t_i \right) H_{\mathbf{Z}}(d\mathbf{t}) \\
&= \int_{\mathbb{W}_+^{d+k}} \left(\bigvee_{i=1}^{d+k} u_i^{\xi} w_i \right) (H_{\mathbf{Y}} \times H_{\mathbf{Z}})(d\mathbf{w}).
\end{aligned}$$

Proof of Lemma 2: Let us show that for any $\mathbf{a} = (a_1, \ldots, a_d)^T > 0$, the max-projection $\vee_{i=1}^d a_i X_i$ is univariate Fréchet. For any positive real number t

$$\begin{aligned}
\mathbb{P}(\vee_{i=1}^d a_i X_i \le t) &= \mathbb{P}\left(Y_1 \vee Z_1 \le t/a_1, \ldots, Y_d \vee Z_d \le t/a_d\right) \\
&= \mathbb{P}(Y_1 \le t/a_1, \ldots, Y_d \le t/a_d, Z_1 \le t/a_1, \ldots, Z_d \le t/a_d) \\
&= \mathbb{P}(V_1 \le t/a_1, \ldots, V_d \le t/a_d, V_{d+1} \le t/a_1, \ldots, V_{2d} \le t/a_d) \\
&= \exp\left[-\sigma_{\mathbf{V}}^{\xi_{\mathbf{V}}}\left(a_1/t, \ldots, a_d/t, a_1/t, \ldots, a_d/t\right)\right] \\
&= \exp\left[-t^{-\xi_{\mathbf{V}}}\sigma_{\mathbf{V}}^{\xi_{\mathbf{V}}}((\mathbf{a}^T; \mathbf{a}^T)^T)\right].
\end{aligned}$$

Hence $\mathbf{X} \sim \mathrm{Fr}(\xi, \mathbf{0}, \sigma_{\mathbf{X}}(\cdot))$ where $\xi_{\mathbf{X}} = \xi_{\mathbf{V}}$ and $\sigma_{\mathbf{X}}(\mathbf{u}) = \sigma_{\mathbf{V}}((\mathbf{u}^T; \mathbf{u}^T)^T)$.
In the case of a discrete spectral measure, one can check that

$$\begin{aligned}
\sigma_{\mathbf{X}}(\mathbf{u}) &= \sigma_{\mathbf{V}}((\mathbf{u}^T; \mathbf{u}^T)^T) = \sum_{j=1}^m \left(\left(\vee_{i=1}^d u_i w_{j,i}\right) \vee \left(\vee_{i=1}^d u_i w_{j,i+d}\right)\right) h_j \\
&= \sum_{j=1}^m \left(\vee_{i=1}^d u_i (w_{j,i} \vee w_{j,i+d})\right) h_j \\
&= \sum_{j=1}^m \left(\vee_{i=1}^d u_i \frac{t_{j,i}}{\|\mathbf{t}_j\|}\right) h_j \|\mathbf{t}_j\| = \sum_{j=1}^m \left(\vee_{i=1}^d u_i \tilde{w}_{j,i}\right) \tilde{h}_j.
\end{aligned}$$

Proof of Lemma 3: To show that $\mathbf{X}$ is a d-dimensional Fréchet random vector, we check that all univariate max-projections are univariate Fréchet. Let $\mathbf{u} \ge 0$ be a d-dimensional vector, and consider the distribution of $\vee_{k=1}^d u_k X_k$. For any positive real number t

$$\begin{aligned}
\mathbb{P}(\vee_{i=1}^d u_i X_i \le t) &= \mathbb{P}(X_i \le t/u_i, i = 1, \ldots, d) = \mathbb{P}(\vee_{j=1}^m a_{ij} Y_j \le t/u_i, i = 1, \ldots, d) \\
&= \mathbb{P}\left(Y_j \le t/(a_{ij} u_i), j = 1, \ldots, m, i = 1, \ldots, d\right) \\
&= \mathbb{P}(Y_j \le t \min_{i=1,\ldots,d}\{1/(a_{ij} u_i)\}, j = 1, \ldots, m) = \mathbb{P}(Y_j \le t / \max_{i=1,\ldots,d}\{a_{ij} u_i\}, j = 1, \ldots, m) \\
&= \exp\left(-\sigma_{\mathbf{Y}}^{\xi}\left(\max_{i=1,\ldots,d}\{a_{i1} u_i\}/t, \ldots, \max_{i=1,\ldots,d}\{a_{im} u_i\}/t\right)\right) \\
&= \exp\left(-t^{-\xi}\sigma_{\mathbf{Y}}^{\xi}(A^T \times_{\max} \mathbf{u})\right).
\end{aligned}$$

This shows that $\mathbf{X}$ is multivariate Fréchet with $\xi_{\mathbf{X}} = \xi$ and $\sigma_{\mathbf{X}}(\mathbf{u}) = \sigma_{\mathbf{Y}}(A^T \times_{\max} \mathbf{u})$. When $H_{\mathbf{Y}}(\cdot)$ is discrete as given in the statement of the Lemma,

$$\begin{aligned}
\sigma_{\mathbf{X}}^{\xi}(\mathbf{u}) &= \sigma_{\mathbf{Y}}^{\xi}(A^T \times_{\max} \mathbf{u}) = \sum_{j=1}^n \left(\vee_{i=1}^m w_{j,i} (\vee_{k=1}^d a_{k,i} u_k)^{\xi}\right) h_j \\
&= \sum_{j=1}^n \left(\vee_{k=1}^d (\vee_{i=1}^m w_{j,i} a_{k,i}^{\xi}) u_k^{\xi}\right) h_j = \sum_{j=1}^n \left(\vee_{k=1}^d t_{j,k} u_k^{\xi}\right) h_j \\
&= \sum_{j=1}^n \left(\vee_{k=1}^d (t_{j,k}/\|\mathbf{t}_j\|) u_k^{\xi}\right) h_j \|\mathbf{t}_j\| = \sum_{j=1}^n \left(\vee_{k=1}^d \widetilde{w}_{j,k} u_k^{\xi}\right) \widetilde{h}_j.
\end{aligned}$$

Proof of Lemma 4: From Lemma 1, we know that if $\mathbf{X}$ is $\mathrm{Fr}(\xi, \boldsymbol{\mu} = \mathbf{0}, H(\cdot))$ then its $\xi-$th power $\mathbf{X}^\star = \mathbf{X}^\xi$ is $\mathrm{Fr}(1, \boldsymbol{\mu} = \mathbf{0}, H(\cdot))$. The spectral measure H doesn't change. One can also say that $\mathbf{X}^\star$ is $\mathrm{Fr}(1, \boldsymbol{\mu} = \mathbf{0}, \sigma^\star(\cdot))$ where the scale function is

$$\sigma^\star(\mathbf{u}) = \sigma^\xi(\mathbf{u}^{1/\xi}).$$

Moreover, the unnormalized Pickands' function on $[0, 1]$ is

$$B^\star(t) = \sigma^\star(1 - t, t) = t \int_0^t (1 - w) dH(w) + (1 - t) \int_t^1 w dH(w).$$

Following the steps of Pickands [1981, Theorem 3.1] or Beirlant et al. [2004, pages 268-269] we have

$$\int_t^1 w dH(w) = \int_t^1 (w-1+1)dH(w) = H((t,1]) - \int_t^1 (1-w)dH(w)$$
$$= \sigma_1^\star + \sigma_2^\star - H([0,t]) - \sigma_2^\star + \int_0^t (1-w)dH(w) = \sigma_1^\star - H([0,t]) + \int_0^t (1-w)dH(w)\ .$$

It yields

$$B(t) = t\int_0^t (1-w)dH(w) + (1-t)\left(\sigma_1^\star - H([0,t]) + \int_0^t (1-w)dH(w)\right)$$
$$= \int_0^t (1-w)dH(w) + (1-t)\left(\sigma_1^\star - H([0,t])\right).$$

Now $\int_0^t (1-w)dH(w) = \int_0^t H([0,u])du + (1-t)H([0,t])$, so that $B^\star(t) = \int_0^t H([0,w])dw + (1-t)\sigma_1^\star$. Then $H([0,t]) = B^{\star\prime}(t) + \sigma_1^\star$, $H(\{0\}) = B^{\star\prime}(0) + \sigma_1^\star$, and $H(\{1\}) = -B^{\star\prime}(1) + \sigma_2^\star$ where $\sigma_i^\star = \sigma^\star(\mathbf{e}_i) = \sigma_i^\xi$ and $B^{\star\prime}$ should be interpreted as its right derivative.

Proof of Lemma 5: Use the transformation $\mathbf{Y} \mapsto \mathbf{Y}^\xi$ and the fact that the corresponding Jacobean is $\prod_{k=1}^d \left(x_k^{(1/\xi)-1}/\xi\right)$.

Proof of Lemma 6: Let $\mathbf{Y} \sim \mathrm{Fr}(\xi_{\mathbf{Y}}, \mu_{\mathbf{Y}}, \sigma_{\mathbf{Y}}(\cdot))$ and $\mathbf{Z} \sim \mathrm{Fr}(\xi_{\mathbf{Z}}, \mu_{\mathbf{Z}}, \sigma_{\mathbf{Z}}(\cdot))$ be independent Fréchet random vectors with discrete spectral measures denoted $H_{\mathbf{Y}}(\cdot)$ and $H_{\mathbf{Z}}(\cdot)$. From Lemma 1 one knows the formula of $H_{\mathbf{X}}$. It is clear that
(a) $H_{\mathbf{X}}(\cdot) = H_{\mathbf{Y}}(\cdot) + H_{\mathbf{Z}}(\cdot)$
(b) $H_{\mathbf{X}}(\cdot) = H_{\mathbf{Y}}(\cdot)$
(c) $H_{\mathbf{X}}(\cdot) = c^{\xi_{\mathbf{Y}}} H_{\mathbf{Y}}(\cdot)$
(f) $H_{\mathbf{X}}(\cdot) = H_{\mathbf{Y}}(\cdot) \times H_{\mathbf{Z}}(\cdot)$
all remain discrete spectral measures on the simplex $\mathbb{W}_+$. The argument for the case (d) is given in details in its statement. Lemmas 2 and 3 give the result in their statements.

Proof of Theorem 1: We will prove that for any vector $\mathbf{a} = (a_1, \ldots, a_d)^T \in \mathbb{R}_+^d \setminus \{\mathbf{0}\}$, the max-projection $\vee_{i=1}^d a_i X_i$ follows a univariate Fréchet distribution. Using the independence of the Z_i's, and the Laplace transform of the positive random vector $\mathbf{S}$ (since $\alpha < 1$), we can write for any positive real number t

$$\begin{aligned}
\mathbb{P}(\vee_{i=1}^d a_i X_i \le t) &= \mathbb{P}(\vee_{i=1}^d a_i S_i^{1/\xi} Z_i \le t) = \mathbb{E}_{\mathbf{S}}\left[\mathbb{P}(Z_1 \le t/(a_1 S_1^{1/\xi}), \ldots, Z_d \le t/(a_d S_d^{1/\xi}))\right] \\
&= \mathbb{E}_{\mathbf{S}}\left[\prod_{i=1}^d \exp\left(-(t/\{a_i S_i^{1/\xi}\})^{-\xi}\right)\right] = \mathbb{E}_{\mathbf{S}}\left[\exp\left(-\sum_{i=1}^d t^{-\xi} a_i^\xi S_i\right)\right] \\
&= \mathbb{E}_{\mathbf{S}}\left[\exp\left(-\langle t^{-\xi}\mathbf{a}^\xi, \mathbf{S}\rangle\right)\right] \ = \exp\left[-c_\alpha t^{-\alpha\xi}\gamma^\alpha(\mathbf{a}^\xi)\right]\ .
\end{aligned}$$

This proves that $\mathbf{X}$ is multivariate Fréchet with $\xi_{\mathbf{X}} = \alpha\xi$ and $\sigma_{\mathbf{X}}^{\xi_{\mathbf{X}}}(\mathbf{a}) = c_\alpha \gamma^\alpha(\mathbf{a}^\xi)$.

Proof of Lemma 7: Let $\mathbf{Y} \sim \mathrm{Fr}(\alpha\xi, \mathbf{0}, \sigma_{\mathbf{Y}}(\cdot))$ and $\mathbf{Z} \sim \mathrm{Fr}(\alpha\xi, \mathbf{0}, \sigma_{\mathbf{Z}}(\cdot))$ be independent Fréchet random vectors constructed as follows. For $U_1, \ldots, U_d, V_1, \ldots, V_d$ independent and identically distributed univariate Fréchet $(\xi, \mu = 0, \sigma = 1)$, we set $\mathbf{Y} = (S_1^{1/\xi} U_1, \ldots, S_d^{1/\xi} U_d)$ and $\mathbf{Z} = (T_1^{1/\xi} V_1, \ldots, T_d^{1/\xi} V_d)$ where $\mathbf{S}$ and $\mathbf{T}$ are positive α-stable random vectors with sum-stable spectral measure $\Lambda_{\mathbf{Y}}$ and $\Lambda_{\mathbf{Z}}$ respectively. One has

$$\sigma_{\mathbf{Y}}^{\alpha\xi}(\mathbf{u}) = c_\alpha \gamma_{\mathbf{Y}}^\alpha(\mathbf{u}^\xi) = c_\alpha \int_{\mathbb{S}_+} \langle \mathbf{u}^\xi, \mathbf{s}\rangle^\alpha \Lambda_{\mathbf{Y}}(d\mathbf{s})$$
$$\sigma_{\mathbf{Z}}^{\alpha\xi}(\mathbf{u}) = c_\alpha \gamma_{\mathbf{Z}}^\alpha(\mathbf{u}^\xi) = c_\alpha \int_{\mathbb{S}_+} \langle \mathbf{u}^\xi, \mathbf{s}\rangle^\alpha \Lambda_{\mathbf{Z}}(d\mathbf{s})\ .$$

The componentwise maximum operation (a) gives $\sigma_{\mathbf{X}}^{\alpha\xi}(\mathbf{u}) = \sigma_{\mathbf{Y}}^{\alpha\xi}(\mathbf{u}) + \sigma_{\mathbf{Z}}^{\alpha\xi}(\mathbf{u}) = c_\alpha \int_{\mathbb{S}_+} \langle \mathbf{u}, \mathbf{s}\rangle^\alpha \{\Lambda_{\mathbf{Y}} + \Lambda_{\mathbf{Z}}\}(d\mathbf{s})$. The power transformation (b) yields $\sigma_{\mathbf{X}}^{\alpha\xi/p}(\mathbf{u}) = \sigma_{\mathbf{Y}}^{\alpha\xi}(\mathbf{u}^{1/p}) = c_\alpha \int_{\mathbb{S}_+} \langle \mathbf{u}^{\xi/p}, \mathbf{s}\rangle^\alpha \Lambda_{\mathbf{Y}}(d\mathbf{s})$. The multiplication by a positive scalar as in (c) implies $\sigma_{\mathbf{X}}^{\alpha\xi}(\mathbf{u}) = c^{\alpha\xi}\sigma_{\mathbf{Y}}^{\alpha\xi}(\mathbf{u}) = c_\alpha \int_{\mathbb{S}_+} \langle \mathbf{u}^\xi, \mathbf{s}\rangle^\alpha \{c^{\alpha\xi}\Lambda_{\mathbf{Y}}\}(d\mathbf{s})$. The componentwise multiplication (d) considers $\mathbf{X} = (c_1 Y_1, ..., c_d Y_d)$ that can be written as $\mathbf{X} = \mathbf{T}^{1/\xi} \cdot \mathbf{U} = (T_1^{1/\xi} U_1, ..., T_d^{1/\xi} U_d)$ where $\mathbf{T} = (c_1^\xi S_1, \dots, c_d^\xi S_d)$. Since $\mathbf{T}$ is a positive α-stable random vector as soon as $\mathbf{S}$ is a positive α-stable random, we conclude that $\mathbf{X}$ is a generalized logistic mixture from its stochastic representation. The multiplication by a sum-stable as in (e) gives

$$\mathbf{X} = S^{1/(\alpha\xi)} \mathbf{Y} = (S^{1/(\alpha\xi)} S_1^{1/\xi} U_1, \dots, S^{1/(\alpha\xi)} S_d^{1/\xi} U_d) = (T_1^{1/\xi} U_1, \dots, T_d^{1/\xi} U_d) \ ,$$

with $\mathbf{T} := S^{1/\alpha} \mathbf{S} = (S^{1/\alpha} S_1, \dots, S^{1/\alpha} S_d)$. Note that if S is positive univariate β-stable random variable and $\mathbf{S}$ is a positive α-stable random vector, with S and $\mathbf{S}$ being independent, then $\mathbf{T}$ is a positive $(\beta\alpha)$-stable random vector. Again the stochastic representation allows to conclude.

Proof of Proposition 2: We use again the differentiation formula

$$\frac{\partial^d}{\partial x_1 \dots \partial x_d} \exp(\phi(\mathbf{x})) = \exp(\phi(\mathbf{x})) \sum_{\pi \in \Pi} \prod_{B \in \pi} \frac{\partial^{|B|} \phi(\mathbf{x})}{\partial^B \mathbf{x}} \ ,$$

with $\phi(\mathbf{x}) = -c_\alpha \gamma^\alpha(\mathbf{x}^{-\xi}) = c_\alpha \int_{\mathbb{S}_+} \langle \mathbf{x}^{-\xi}, \mathbf{s} \rangle^\alpha \Lambda(d\mathbf{s})$ so that

$$\frac{\partial^{|B|} \phi(\mathbf{x})}{\partial^B \mathbf{x}} = (-1)^{1+|B|} \frac{\partial^{|B|} I(\mathbf{x})}{\partial^B \mathbf{x}}$$

which allows to conclude since $\prod_{B \in \pi} (-1)^{1+|B|} = (-1)^{|\pi|+d}$.

Proof of Proposition 3: We apply Lemma 4. In this setting, we have

$$\sigma^\star(\mathbf{u}) = \sigma^{\alpha\xi}(\mathbf{u}^{1/(\alpha\xi)}) = c_\alpha \gamma^\alpha(\mathbf{u}^{1/\alpha})$$

and $\sigma_i^\star = c_\alpha \int_{\mathbb{S}_+} s_i^\alpha \Lambda(d\mathbf{s})$ for $i = 1, 2$. We get successively

$$B^\star(t) = \sigma^\star(1-t, t) = c_\alpha \gamma^\alpha((1-t)^{1/\alpha}, t^{1/\alpha}),$$

$$B^{\star\prime}(t) = c_\alpha \int_{\mathbb{S}_+} \left((1-t)^{1/\alpha} s_1 + t^{1/\alpha} s_2\right)^{\alpha-1} \left(-(1-t)^{1/\alpha-1} s_1 + t^{1/\alpha-1} s_2\right) \Lambda(d\mathbf{s}),$$

and $h(t) = c_\alpha(1/\alpha - 1)(t(1-t))^{1/\alpha-2} \int_{\mathbb{S}_+} ((1-t)^{1/\alpha} s_1 + t^{1/\alpha} s_2)^{\alpha-2} s_1 s_2 \Lambda(d\mathbf{s})$.

Proof of Lemma 8: The kernel $k(u, w) = (1-u)w \vee u(1-w)$ is always a V-shaped function with vertex at $u = w$. In particular, if $u \leq a$, then $k(u, w) = (1-u)w$ and

$$g_p(u; a, b) = \int_a^b (1-u) w w^p dw = (1-u) \int_a^b w^{p+1} dw = (1-u) c_1 \ ,$$

where $c_1 = (b^{p+2} - a^{p+2})/(p+2)$.

Likewise, if $u \geq b$, then $k(u, w) = u(1-w)$ and $g_p(u; a, b) = \int_a^b u(1-w) w^p dw = u c_5$, where $c_5 = (b^{p+1} - a^{p+1})/(p+1) - (b^{p+2} - a^{p+2})/(p+2)$.

For $a < u < b$, the kernel $k(u, w) = u(1-w)$ on (a, u), while $k(u, w) = (1-u)w$ on (u, b). Hence for $u \in (a, b)$,

$$\begin{aligned} g_p(u; a, b) &= \int_a^u u(1-w) w^p dw + \int_u^b (1-u) w w^p dw \\ &= u \left(\frac{w^{p+1}}{p+1} - \frac{w^{p+2}}{p+2} \right) \Big|_a^u + (1-u) \left(\frac{w^{p+2}}{p+2} \right) \Big|_u^b \\ &= u^{p+3} \left(\frac{1}{p+2} - \frac{1}{p+2} \right) + u^{p+2} \left(\frac{1}{p+1} - \frac{1}{p+2} \right) + u \left(-\frac{b^{p+2}}{p+2} - \frac{a^{p+1}}{p+1} + \frac{a^{p+2}}{p+2} \right) + \frac{b^{p+2}}{p+2} \end{aligned}$$

so that $g_p(u; a, b) = c_2 u^{p+2} - c_3 u + c_4$.

Proof of Lemma 9: Let $\mathbf{Y} \sim \text{Fr}(\xi_\mathbf{Y}, \mu_\mathbf{Y}, \sigma_\mathbf{Y}(\cdot))$ and $\mathbf{Z} \sim \text{Fr}(\xi_\mathbf{Z}, \mu_\mathbf{Z}, \sigma_\mathbf{Z}(\cdot))$ be independent Fréchet random vectors with piecewise polynomial spectral densities denoted $h_\mathbf{Y}(\mathbf{w}) = \sum_{j=1}^{m_\mathbf{Y}} p_{\mathbf{Y},j}(\mathbf{w}) 1_{\Delta_{\mathbf{Y},j}}(\mathbf{w})$ and

$h_{\mathbf{Z}}(\mathbf{w}) = \sum_{j=1}^{m_{\mathbf{Z}}} p_{\mathbf{Z},j}(\mathbf{w}) 1_{\Delta_{\mathbf{Z},j}}(\mathbf{w})$. From Lemma 1 one knows the formula of $H_{\mathbf{X}}$. It is clear that
(a) $h_{\mathbf{X}}(\cdot) = h_{\mathbf{Y}}(\cdot) + h_{\mathbf{Z}}(\cdot)$
(b) $h_{\mathbf{X}}(\cdot) = h_{\mathbf{Y}}(\cdot)$
(c) $h_{\mathbf{X}}(\cdot) = c^{\xi_{\mathbf{Y}}} h_{\mathbf{Y}}(\cdot)$
(d) $h_{\mathbf{X}}(\mathbf{v}) = h_{\mathbf{Y}}(\mathbf{v}\mathbf{c}^{-\xi}/\|\mathbf{v}\mathbf{c}^{-\xi}\|)\, \|\mathbf{v}\mathbf{c}^{-\xi}\|^{-(d+1)} \prod_{i=1}^{d} c_i^{-\xi}$
(f) $h_{\mathbf{X}} = h_{\mathbf{Y}} \times h_{\mathbf{Z}}$
all remain piecewise polynomial spectral densities on the simplex $\mathbb{W}_+$.
Proof of Theorem 2: For any $\mathbf{x} \in \mathbb{R}^d$,

$$\begin{aligned} |G_{\mathbf{X}}(\mathbf{x}) - G_{\mathbf{Y}}(\mathbf{x})| &= \left|\exp(-\sigma_{\mathbf{X}}^{\xi}(\mathbf{x}^{-1})) - \exp(-\sigma_{\mathbf{Y}}^{\xi}(\mathbf{x}^{-1}))\right| \\ &= \exp(-\sigma_{\mathbf{X}}^{\xi}(\mathbf{x}^{-1})) \left|1 - \exp(-\left[\sigma_{\mathbf{Y}}^{\xi}(\mathbf{x}^{-1}) - \sigma_{\mathbf{X}}^{\xi}(\mathbf{x}^{-1})\right])\right| \\ &\le \exp(-\sigma_0^{\xi}\|\mathbf{x}^{-1}\|^{\xi}) \left|1 - \exp\left(-\|\mathbf{x}^{-1}\|^{\xi}\left[\sigma_{\mathbf{Y}}^{\xi}(\mathbf{x}^{-1}/\|\mathbf{x}^{-1}\|) - \sigma_{\mathbf{X}}^{\xi}(\mathbf{x}^{-1}/\|\mathbf{x}^{-1}\|)\right]\right)\right| \\ &\le \exp(-\sigma_0^{\xi}\|\mathbf{x}^{-1}\|^{\xi}) \max\{\exp(\delta\|\mathbf{x}^{-1}\|^{\xi}) - 1, 1 - \exp(-\delta\|\mathbf{x}^{-1}\|^{\xi})\} \\ &\le \exp(-\sigma_0^{\xi}\|\mathbf{x}^{-1}\|^{\xi}) \left(\exp(\delta\|\mathbf{x}^{-1}\|^{\xi}) - \exp(-\delta\|\mathbf{x}^{-1}\|^{\xi})\right). \end{aligned}$$

Some calculus shows that $g(t) := e^{-\sigma_0^{\xi} t}(e^{\delta t} - e^{-\delta t})$ has a maximum at $t^\star = 1/(2\delta)\ln((\sigma_0^{\xi}+\delta)/(\sigma_0^{\xi}-\delta))$, and $g(t^\star) = \frac{2\delta}{\sigma_0^{\xi}-\delta}\left(\frac{\sigma_0^{\xi}+\delta}{\sigma_0^{\xi}-\delta}\right)^{-(\sigma_0^{\xi}+\delta)/(2\delta)} \le (2\delta)/\sigma_0^{\xi}$. Applying this to the bound above gives the result.

Proof of Theorem 3: The kernel function $k(\mathbf{w}, \mathbf{u}) = \vee_{i=1}^{d} u_i^{\xi} w_i$ is Hölder continuous in the first variable: $|k(\mathbf{w}_1, \mathbf{u}) - k(\mathbf{w}_2, \mathbf{u})| \le K^2 \|\mathbf{w}_1 - \mathbf{w}_2\|$ uniformly in $\mathbf{u} \in \mathbb{W}_+$. Therefore, Lemma 4.2 of Nolan [2010] implies $|\sigma_{\mathbf{X}}^{\xi}(\mathbf{u}) - \sigma_{\mathbf{Y}}^{\xi}(\mathbf{u})| \le K^2(1+K^2)\pi^*(H_{\mathbf{X}}, H_{\mathbf{Y}}) \le K^2(1+K^2)\delta$ for any $\mathbf{u} \in \mathbb{W}_+$, where the last inequality follows from the assumption. Applying Theorem 2 finishes the proof.

Proof of Proposition 4: Let the simplex $\mathbb{W}_+ \subset \mathbb{R}_+^d$ be partitioned into uniformly small pieces $\Delta_1, \ldots, \Delta_m$, e.g. satisfying $\mathrm{Vol}_{d-1}(\Delta_j) = \mathrm{Vol}_{d-1}(\mathbb{W}_+)/m$. Let $\mathbf{p}_j$ be a "midpoint" of Δ_j, and define

$$H_{\mathrm{disc},m}(\cdot) = \sum_{j=1}^{m} H(\Delta_j)\delta_{\mathbf{p}_j}(\cdot).$$

This measure will be close to the original spectral measure H in the extended Prokorov metric π^*, so applying Theorem 3 gives the result.

Proof of Proposition 5: We already know from Proposition 4 that the discrete spectral measures give a dense class: for any positive ϵ, there exists G_{disc} such that $|G - G_{\mathrm{disc}}| < \epsilon$. If σ_{disc} denotes its associated scale function, then it can be written as

$$\sigma_{\mathrm{disc}}(\mathbf{u}) = \left(\sum_{j=1}^{m} \sigma_{j,\mathrm{disc}}(\mathbf{u}^{\xi})\right)^{1/\xi}$$

with $\sigma_{j,\mathrm{disc}}(\mathbf{u}) = \vee_{i=1}^{d} h_j w_{j,i} u_i$; see the first lines of Section 3.1. In terms of random vectors, what precedes can be expressed as follows: $\mathbf{X} \sim \mathrm{Fr}(\xi, \mathbf{0}, \sigma_{\mathrm{disc}})$ can be generated by

$$\mathbf{X} = \max\{\mathbf{Y}_1^{1/\xi}, \ldots, \mathbf{Y}_m^{1/\xi}\}$$

with $\mathbf{Y}_j \sim \mathrm{Fr}(1, \mathbf{0}, \sigma_{j,\mathrm{disc}})$. As a consequence, the result of Proposition 5 will follow from Theorem 2 as soon as there exists a scale function $\sigma_{j,\log}$ from a generalized logistic mixture close enough to $\sigma_{j,\mathrm{disc}}$. Indeed, the inverse operation will use the fact that the class of generalized logistic distributions is closed under the previous transformations, as stated in Lemma 7. One can check that such a scale function exists since $\sigma_{j,\log}(\mathbf{u}) := (\sum_{i=1}^{d} (h_j w_{j,i} u_i)^{\alpha})^{1/\alpha}$ converges to $\vee_{i=1}^{d} h_j w_{j,i} u_i$ as $\alpha \downarrow 0$.

Proof of Proposition 6: Consider a fine partition $\Delta_1, \ldots, \Delta_m$ of $\mathbb{W}_+$ as already defined in the proof of Proposition 4. Let h_{pp} be the piecewise constant spectral density defined by

$$h_{\mathrm{pp}}(\mathbf{w}) = \sum_{j=1}^{m} H(\Delta_j)\mathbf{1}_{\Delta_j}(\mathbf{w}).$$

This density corresponds to an spectral measure H_{pp} for which $\pi^\star(H, H_{\mathrm{pp}})$ is arbitrarily small.

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